THEOREM 1.12. *The theta series $\Theta(z;P)$ is an holomorphic function on $\mathbf{H}$ satisfying the following automorphy relations: for any $\gamma\in\Gamma_0(4)$,*

$$\Theta(\gamma z;P)=((\frac{c}{d})\varepsilon_d^{-1})^l(cz+d)^{l/2+d}\Theta(z;P).$$

We will say that $\Theta(\gamma z;P)$ is an holomorphic modular form of weight

$$k=l/2+d.$$

5.4.1. *Proof of Theorem 1.12.* The proof which we leave to the reader uses crucially the following multidimensional version of the Poisson summation formula which is proven by multiple application of the one dimensional Poisson formula:

THEOREM 1.13. *Let $f\in\mathcal{S}(\mathbf{R}^l)$ be in the Schwarz class, for $u\in\mathbf{R}^l$, one has*

$$\sum_{\mathbf{x}\in\mathbf{Z}^l}f(\mathbf{x}+u)=\sum_{\mathbf{x}'\in\mathbf{Z}^l}\widehat{f}(\mathbf{x}')e(-\mathbf{x}'.u)$$

where $\mathbf{x}.u=\sum_{i=1}^l x_i'u_i$ denote the Euclidean inner product and

$$\widehat{f}(\mathbf{x}')=\int_{\mathbf{R}^l}f(\mathbf{x})e(\mathbf{x}.\mathbf{x}')d\mathbf{x}$$

is the Fourier transform.

In particular one need the following

THEOREM 1.14. *Let $P\in\mathcal{H}_{l,d}$ and*

$$f_P(\mathbf{x})=P(\mathbf{x})e^{-\pi Q_l(\mathbf{x})}=P(\mathbf{x})e^{-\pi(\sum_i x_i^2)}$$

then

$$\widehat{f_P}(\mathbf{x}')=i^{-d}f_P(\mathbf{x}').$$

PROOF. When $P=1$,

$$\hat{f}_1(\mathbf{x}')=\int_{\mathbf{R}^l}\prod_i e^{-\pi x_i^2}e(x_i.x_i')dx_i=\prod_i\widehat{e^{-\pi x_i^2}}(x_i')=\prod_i e^{-\pi x'^2_i}=f_1(\mathbf{x}').$$

Take $P(\mathbf{x})=(\mathbf{c}.\mathbf{x})^d$ and let

$$L=L_{\mathbf{c}}=\sum_i c_i\frac{\partial}{\partial x_i}$$

We have by integration by parts and for any $f\in\mathcal{S}(\mathbf{R}^l)$

$$\widehat{\mathbf{c}.\mathbf{x}f}(\mathbf{x}')=-\frac{1}{2\pi i}L_c\widehat{f}(\mathbf{x}')$$

and

$$\widehat{P_{\mathbf{c}}(\mathbf{x})}f(\mathbf{x}')=(\widehat{\mathbf{c}.\mathbf{x})^d}f(\mathbf{x}')=\frac{(-1)^d}{(2\pi i)^d}L_c^d(\widehat{f})(\mathbf{x}')$$

taking $f(\mathbf{x})=f_1(\mathbf{x})$ in this identity we conclude since one proves by recurrence that

$$L_c^df_1=(-2\pi)^dP_{c,d}(\mathbf{x})f_1(\mathbf{x})=(-2\pi)^df_P(\mathbf{x}).$$

Indeed

$$L_ce^{-\pi(\sum_i x_i^2)}=\sum_i-2\pi c_ix_if_1(\mathbf{x})=(-2\pi)P_{c,1}(\mathbf{x})f_1(\mathbf{x}).$$

Suppose that
$$L_c^d f_1 = (-2\pi)^d P_{c,d}(\mathbf{x}) f_1(\mathbf{x}),$$
we have
$$L_c^{d+1} f_1 = (-2\pi)^d [L_c(P_{c,d}) f_1(\mathbf{x}) + P_{c,d} L_c f_1] = (-2\pi)^d [L_c(P_{c,d}) f_1(\mathbf{x}) + (-2\pi) P_{c,d+1} f_1],$$
Now
$$L_c(P_{c,d}) = L_c((\mathbf{c}.\mathbf{x})^d) = d(\mathbf{c}.\mathbf{x})^{d-1} L_c(\mathbf{c}.\mathbf{x}) = d(\mathbf{c}.\mathbf{x})^{d-1} \sum_i c_i^2 = 0.$$
□

5.4.2. *Proof of Theorem 5.5.* As we will show later, the fact that P is not constant along with the automorphy relations satisfied by $\Theta(z; P)$, implies that the function
$$y^{k/2} |\Theta(z; P)|$$
is *bounded* on $\mathbf{H}$; this implies that there exists $\delta \geqslant 0$ such that
$$r_l(n; P) \ll n^{(l/2+d)/2-\delta}. \tag{5.6}$$
Hence by Proposition 1.1 we have
$$\frac{1}{r_l(n)} \sum_{Q_l(\mathbf{x})=n} P(\frac{\mathbf{x}}{\sqrt{n}}) \ll n^{-l/4+1-\delta+o(1)}.$$
If $l \geqslant 5$ this converge to 0 as $n \to +\infty$ since $\delta \geqslant 0$. One can in fact prove (but this is harder) that $\delta > 0$ so the above sum also converge to 0 for $l = 4$, $n \to +\infty$ (and odd).

In order to prove (5.6) we observe that for any $y > 0$
$$\exp(-2\pi n y) y^{k/2} r_l(n; P) = \int_{[0,1]} y^{k/2} \Theta(x + iy) e(-nx) dx = O(1).$$
Taking $y = 1/n$ we obtain
$$r_l(n; P) \ll n^{k/2}.$$

5.5. Explanation of the figures. The pictures of figure 2 represent the distribution of the sets
$$\frac{1}{\sqrt{5^\alpha}} \mathcal{R}_4(5^\alpha)$$
on the 3-dimensional sphere $S_4 \subset \mathbf{R}^4$. It is of course no simple to represent the 3-sphere on a two dimensional plane and here we use the *Hopf map* trick: let us recall that $\mathbf{R}^4$ may be identified with the (associative, non-commutative) algebra of Hamilton quaternions $\mathbb{H}$
$$(x, y, z, t) \to w = x + iy + jz + kt, \quad \text{where } i^2 = j^2 = k^2 = -1,\ ij = -ji = k.$$
Recall that $\mathbb{H}$ is equipped with a canonical involution, a reduced trace and a reduced norm:
$$w = x+iy+jz+kt \to \overline{w} = x-iy-jz-kt, \mathrm{Tr}_{\mathbb{H}}(w) = w+\overline{w} = 2x,\ \mathrm{Nr}_{\mathbb{H}}(w) = w\overline{w} = x^2+y^2+z^2+t^2.$$
Therefore the map
$$(x, y, z, t) \to w = x + iy + jz + kt$$
is an isometry between the quadratic spaces $(\mathbf{R}^4, Q_4)$ and $(\mathbb{H}, \mathrm{Nr}_{\mathbb{H}})$: ie. the square of the Euclidean norm correspond to the Quaternionic norm:
$$x^2 + y^2 + z^2 + t^2 = \mathrm{Nr}_{\mathbb{H}}(x + iy + jz + kt) = (x + iy + jz + kt)(x - iy - jz - kt).$$
Therefore S_4 gets identified with the quaternions of norm 1,
$$\mathbb{H}^1 = \{w \in \mathbb{H},\ \mathrm{Nr}_{\mathbb{H}}(w) = 1\},$$

which form a group under multiplication (since the norm is multiplicative

$$\mathrm{Nr}_{\mathbb{H}}(ww') = \mathrm{Nr}_{\mathbb{H}}(w)\mathrm{Nr}_{\mathbb{H}}(w')\ .)$$

Next the Euclidean 3-space $(\mathbf{R}^3, Q_3)$ is isometrically identified with the subspace of pure quaternions $(\mathbb{H}^0, \mathrm{Nr}_{\mathbb{H}})$ via

$$(y, z, t) \in \mathbf{R}^3 \to iy + jz + kt \in \mathbb{H}^0.$$

The linear action of $\mathbb{H}^1$ on $\mathbb{H}$ by conjugation, given for $w \in \mathbb{H}^1$ by

$$\rho_w :\ w' \in \mathbb{H} \to ww'w^{-1}$$

preserve the norm (ie. $\mathrm{Nr}_{\mathbb{H}}(ww'w^{-1}) = \mathrm{Nr}_{\mathbb{H}}(w')$) as well as the trace (ie. $\mathrm{Tr}_{\mathbb{H}}(ww'w^{-1}) = \mathrm{Tr}_{\mathbb{H}}(w')$) and leave $\mathbb{H}^0$ invariant. Therefore under the identification $\mathbb{H}^0 \simeq \mathbf{R}^3$, ρ_w correspond to an isometry[2] which is in fact of determinant $+1$. Therefore we have a group homomorphism, the *Hopf map*

$$\mathrm{Hopf} : S_4 \simeq \mathbb{H}^1 \to \mathrm{SO}_3(\mathbf{R}).$$

This map is in fact surjective and its kernel is $\{(\pm 1, 0, 0, 0)\} \simeq \{\pm 1_{\mathbb{H}}\}$.

Thus to any $\mathbf{x} \in \mathcal{R}_4(n)$ one associate the isometry $\mathrm{Hopf}(\frac{1}{\sqrt{n}}\mathbf{x}) \in \mathrm{SO}_3(\mathbf{R})$. In the pictures we have represented the various images of the vector $(1, 0, 0)$ of S_3 under the rotations

$$\mathrm{Hopf}(\frac{1}{\sqrt{n}}\mathbf{x}),\ \mathbf{x} \in \mathcal{R}_4(n),\ n = 5^2, 5^3, 5^4.$$

These points seem and in fact are equidistributed on S_3 with respect to the natural $\mathrm{SO}_3(\mathbf{R})$-invariant measure μ_3 as n gets large: this is a consequence of the fact that the set $\frac{1}{\sqrt{n}}\mathcal{R}_4(n)$ are equidistributed on S_4 wrt μ_4 as $n \to +\infty$.

This last interpretation yields to a further generalization of the theta series:

6. Theta series associated to general definite quadratic forms

Let

$$Q(\mathbf{x}) = \sum_i a_{ii}x_i^2 + \sum_{i<j} 2a_{ij}x_i x_y = \sum_{i,j} a_{ij}x_i x_j,\ a_{ij} = a_{ji}$$

be a non-degenerate quadratic form on $\mathbf{R}^l$. We denote by

$$\langle \mathbf{x}, \mathbf{x}' \rangle_Q = \frac{1}{2}(Q(\mathbf{x} + \mathbf{x}') - Q(\mathbf{x}) - Q(\mathbf{x}'))$$

the associated inner product (or polarization). Let

$$A = (a_{ij})_{ij} = (\langle e_i, e_j \rangle_Q)_{i,j}$$

be the associated symmetric matrix; the assumption that Q is non-degenerate is equivalent to $\det A \neq 0$. We assume that Q is positive and A is integral; in particular $Q(\mathbf{Z}^l) \subset \mathbf{Z}$.

Let

$$\mathcal{R}_Q(n) = \{\mathbf{x} \in \mathbf{Z}^l,\ Q(\mathbf{x}) = n\},\ r_Q(n) := |\mathcal{R}_Q(n)|$$

We form

$$\Theta_Q(z) = \sum_{\mathbf{x} \in \mathbf{Z}^l} e(Q(\mathbf{x})z) = \sum_{n \geqslant 0} r_Q(n) e(nz)$$

[2] This is either the identity if $w \in \mathbf{R}.1$ is a scalar or the symetry whose axis is the line generated by the trace 0-vector $w_0 = w - \mathrm{Tr}_{\mathbb{H}}(w)$

where

$$r_Q(n) = |r_Q(n)|,\ \mathcal{R}_Q(n) = \{\mathbf{x} \in \mathbf{Z}^l,\ Q(\mathbf{x}) = n\}$$

is the cardinality of the set of integral representations of n by Q.

It is easy to see that $r_Q(n) \ll n^{l/2}$ so the series converge rapidly and define an holomorphic function on $\mathbf{H}$

It turn out that this theta function satisfies automorphy relations similar to the theta function associated to the Euclidean quadratic form $\Theta_l(z) = \Theta(z)^l$ but for a smaller subgroup of $\mathrm{SL}_2(\mathbf{Z})$: for $q \geqslant 1$ let

$$\Gamma_0(q) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ c \equiv 0(q)\}$$

be the Iwahori-Hecke subgroup of level q. One can prove the following

THEOREM 1.15. *Let N be such that NA^{-1} is integral, then $\Theta_Q(z)$ is a holomorphic function on $\mathbf{H}$ satisfying. For any $\gamma \in \Gamma_0(4N)$*

$$\Theta_Q(\gamma z) = (\frac{\det A}{d})((\frac{c}{d})\varepsilon_d^{-1})^l(cz+d)^{l/2}\Theta(z).$$

The proof use the following variant of the Poisson summation formula; for this we need first to recall the notion of dual lattice and dual basis: if $\{f_1, \cdots, f_l\}$ is a basis of $\mathbf{R}^l$, its dual basis with respect to Q is the basis

$$\{f_1^*, \cdots, f_l^*\},\ \textit{such that}\ \langle f_i, f_j^*\rangle = \delta_{ij}.$$

Correspondingly let

$$L = \mathbf{Z}f_1 + \cdots + \mathbf{Z}f_l$$

be the lattice generated by this basis the dual lattice is

$$L^* = \mathbf{Z}f_1^* + \cdots + \mathbf{Z}f_l^* = \{\mathbf{x}' \in \mathbf{R}^l,\ \forall \mathbf{x} \in L\ \langle \mathbf{x}, \mathbf{x}'\rangle_Q \in \mathbf{Z}\}$$

THEOREM 1.16. *Let $f \in \mathcal{S}(\mathbf{R}^l)$ be in the Schwarz class, for $u \in \mathbf{R}^l$, one has*

$$\sum_{\mathbf{x}\in\mathbf{Z}^l} f(\mathbf{x}+u) = \sum_{\mathbf{x}^*\in(\mathbf{Z}^l)^*} \widehat{f}(\mathbf{x}^*)e(-\langle \mathbf{x}^*, u\rangle_Q)$$

where

$$\widehat{f}(\mathbf{x}') = \int_{\mathbf{R}^l} f(\mathbf{x})e(\langle \mathbf{x}, \mathbf{x}'\rangle_Q)d\mathbf{x}$$

denote the Fourier transform relative to the inner product $\langle ., .\rangle_Q$.

6.1. Harmonic polynomials associated to a quadratic form. Let $A^{-1} = (a_{ij}^*)$ the Laplace operator with respect to Q is defined as

$$\Delta_Q = \sum_{i,j} a_{ij}^* \frac{\partial^2}{\partial x_i \partial x_j}$$

Let P be an harmonic polynomial of degree d with respect to Q (ie. $\Delta_Q(P) = 0$), then we form

$$\Theta_Q(z; P) = \sum_{\mathbf{x}\in\mathbf{Z}^l} P(\mathbf{x})e(Q(\mathbf{x})z) = \sum_{n\geqslant 0} r_Q(n; P)e(nz)$$

THEOREM 1.17. *Let N be such that NA^{-1} is integral, then $\Theta_Q(z;P)$ is a holomorphic function on $\mathbf{H}$ satisfying, for any $\gamma\in\Gamma_0(4N)$*

$$\Theta_Q(\gamma z;P)=(\frac{\det A}{d})((\frac{c}{d})\varepsilon_d^{-1})^l(cz+d)^{l/2+d}\Theta(z;P).$$

6.2. Equidistribution of representations. One can prove the following (which is not easy):

THEOREM 1.18. *For $l\geqslant 4$ there exists explicit(able) integers $a,q>0$ such that for $n\equiv a(q)$ $(n>0)$*

$$r_Q(n)=n^{l/2-1+o(1)},\ n\rightarrow+\infty.$$

Moreover one has also an equidistribution statement: let

$$S_Q=\{\mathbf{x}\in\mathbf{R}^l,\ Q(\mathbf{x})=1\}$$

be variety of level 1 associated to the quadratic form Q: this is an ellipsoid and also the "unit sphere" for the inner product induced by Q:

$$\langle\mathbf{x},\mathbf{x}'\rangle_Q:=\frac{1}{2}(Q(\mathbf{x}+\mathbf{x}')-Q(\mathbf{x})-Q(\mathbf{x}')).$$

As for the usual sphere S_Q carries a unique probability measure which is $\mathrm{SO}_Q(\mathbf{R})$-invariant, μ_Q say, then we have

THEOREM 1.19. *Let a,q be as above and $f\in\mathcal{C}(S_Q)$; for $n\equiv a(q)$, one has*

$$\frac{1}{r_Q(n)}\sum_{\mathbf{x}\in\mathcal{R}_Q(n)}f(\frac{\mathbf{x}}{n^{1/2}})\rightarrow\mu_Q(f),\ n\rightarrow+\infty.$$

CHAPTER 2

The Upper-half plane

In the previous chapter we have seen examples on functions (the theta functions) which are holomorphic functions on the upper-half plane $\mathbf{H}$ satisfying some sort of invariance properties with respect to the action of some subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})\mathrm{SL}_2(\mathbf{R})$ which is induced by the action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathbf{H}$ by fractional linear transformations. In this chapter we describe structural aspect of this action and introduce some elements of hyperbolic geometry.

1. The complex projective line

The projective line $\mathrm{P}^1(\mathbf{C})$ is by definition the set of lines in $\mathbf{C}^2$ passing through the origin $(0,0)$:

$$\mathrm{P}^1(\mathbf{C}) = \{L \subset \mathbf{C}^2, 0 \in L \text{ a line}\}.$$

The lines passing through the origin are parametrized by their *slope* $z \in \mathbf{C} \cup \{\infty\}$

$$z \in \mathbf{C},\ L_z : X = zY,\ L_\infty : 0 = Y,$$

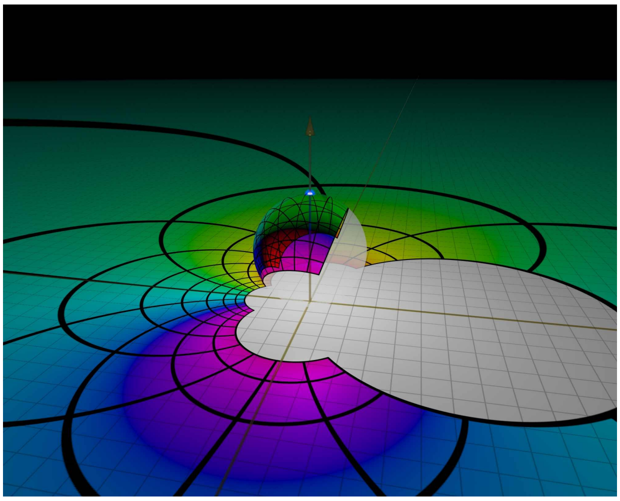

FIGURE 1. Moebius trasformation revealed: a movie by D. Arnold and J. Rognes.

so we identify $\mathrm{P}^1(\mathbf{C})$ with $\mathbf{C}\cup\{\infty\}$.

2. Fractional linear transformations

The general linear group $\mathrm{GL}_2(\mathbf{C})$ acts on $\mathbf{C}^2$ by linear transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX+bY \\ cX+dY \end{pmatrix}.$$

In particular, its acts on the space of lines passing through the origin $\mathrm{P}^1(\mathbf{C})$ and in terms of the slope parametrization this action is given by

$$g(L_z) = L_{gz},\ g.z = \frac{az+b}{cz+d},$$

with the convention that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\infty = \frac{a}{c},\ c\neq 0,\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\infty = \infty.$$

Observe that the group of scalar matrices

$$Z(\mathbf{C}) = \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},\ a\in\mathbf{C}^\times\} = \mathbf{C}^\times\mathrm{Id}$$

act trivially: it follows that we have an action by the quotient group

$$\mathrm{PGL}_2(\mathbf{C}) = \mathrm{GL}_2(\mathbf{C})/\mathbf{C}^\times\mathrm{Id}$$

which is called the *projective linear group*. Alternatively we may always replace a matrix g of $\mathrm{GL}_2(\mathbf{C})$ by a matrix of determinant 1 by multiplying g by the scalar matrix $\det g^{-1/2}\mathrm{Id}$, so without loss of information we may restrict to the action of *the special linear group* $\mathrm{SL}_2(\mathbf{C})$ or to the *projective special linear group*

$$\mathrm{PSL}_2(\mathbf{C}) = \mathrm{SL}_2(\mathbf{C})/\{\pm\mathrm{Id}\}.$$

2.1. Orbits. The structure of the space of orbits $\mathrm{GL}_2(\mathbf{C})\backslash\mathrm{P}^1(\mathbf{C})$ is very simple: there is only one orbit (the group $\mathrm{GL}_2(\mathbf{C})$ acts transitively on $\mathrm{P}^1(\mathbf{C})$): for $c\neq 0$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\infty = \frac{a}{c},$$

so for any $z\in\mathrm{P}^1(\mathbf{C})$,

$$\mathrm{P}^1(\mathbf{C}) = \mathrm{GL}_2(\mathbf{C}).\infty = \mathrm{GL}_2(\mathbf{C}).z.$$

The stabilizer of ∞ is the *Borel subgroup* (of upper-triangular matrices)

$$\mathrm{GL}_2(\mathbf{C})_\infty = B(\mathbf{C}) = \{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\in\mathrm{GL}_2(\mathbf{C})\}$$

and hence

$$\mathrm{P}^1(\mathbf{C}) \simeq \mathrm{GL}_2(\mathbf{C})/B(\mathbf{C}),$$

given by the inverse of the map $gB(\mathbf{C})\mapsto g.\infty$.

2.2. Fixed points. Given $g \in \mathrm{GL}_2(\mathbf{C})$, a fixed point of g is a $z \in \mathrm{P}^1(\mathbf{C})$ such that

$$g.z = z.$$

These can be determined by solving the equation of degree $\leqslant 2$

$$az + b = z(cz + d) \Longleftrightarrow cz^2 + (d - a)z - b = 0$$

and either the set of fixed point is the whole of $\mathrm{P}^1(\mathbf{C})$ (if and only if $g = \lambda\mathrm{Id}$, $\lambda \in \mathbf{C}^\times$) or has at most 2 elements. If fact, thinking in term of lines in $\mathbf{C}^2$, one see that z is a fixed point of g if and only if the line L_z is an eigenspace for the linear map g on $\mathbf{C}^2$. This depend whether g is, up to multiplication by a scalar, the identity, a unipotent matrix (different from the identity), or a non-unipotent matrix.

2.3. The Bruhat decomposition. The Borel subgroup decompose further as

$$B(\mathbf{C}) = N(\mathbf{C})D(\mathbf{C}) = Z(\mathbf{C})N(\mathbf{C})A(\mathbf{C}),$$

where

$$N(\mathbf{C}) = \{n(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},\ z \in \mathbf{C}\},$$

is the group of upper-triangular unipotent matrices and

$$D(\mathbf{C}) = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},\ a, d \in \mathbf{C}^\times\},\ A(\mathbf{C}) = D(\mathbf{C}) \cap \mathrm{SL}_2(\mathbf{C}) = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix},\ a \in \mathbf{C}^\times\}$$

are the groups of diagonal matrices. We now prove some further decompositions: consider the inversion matrix

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then $w^2 = -\mathrm{Id}$ so w is an involution on $\mathrm{P}^1(\mathbf{C})$ with $\pm i$ as fixed points,

$$wz = -1/z,\ w^2 z = z,\ w.0 = \infty,\ w.i = i.$$

PROPOSITION 2.1 (Bruhat decomposition). *One has*

$$\mathrm{GL}_2(\mathbf{C}) = B(\mathbf{C}) \sqcup N(\mathbf{C})wB(\mathbf{C})$$

and this decomposition is unique (a matrix g not in $B(\mathbf{C})$ can be written in a unique way into to form $g = nwb$ with $n \in N$ and $b \in B$). Similarly one has

$$\mathrm{SL}_2(\mathbf{C}) = N(\mathbf{C})A(\mathbf{C}) \sqcup N(\mathbf{C})wN(\mathbf{C})A(\mathbf{C})$$

PROOF. One has

$$\mathrm{P}^1(\mathbf{C}) = \{\infty\} \sqcup \mathbf{C}$$

moreover, since N acts by translations

$$n(z)z' = z + z',$$

$\mathbf{C}$ is the N-orbit of $0 = w\infty$ (moreover $N(\mathbf{C})_0 = \{\mathrm{Id}\}$)

$$\mathbf{C} = N(\mathbf{C}).0 = N(\mathbf{C})w\infty$$

since $B(\mathbf{C})$ is the stabilizer of ∞ in $\mathrm{GL}_2(\mathbf{C})$,

$$\mathrm{GL}_2(\mathbf{C})/B(\mathbf{C}) \simeq \mathrm{P}^1(\mathbf{C}) \simeq \{\infty\} \sqcup \mathbf{C} \simeq B(\mathbf{C})/B(\mathbf{C}) \sqcup N(\mathbf{C})wB(\mathbf{C})/B(\mathbf{C}).$$

This decomposition is unique: if g is not in $B(\mathbf{C})$ and satisfies $g = nwb = n'wb'$ then

$$g\infty = nw\infty = n.0 = n'.0 \Rightarrow n = n'$$

and then $b = b'$. □

Since $B(\mathbf{C}) = Z(\mathbf{C})N(\mathbf{C})A(\mathbf{C})$ one has

COROLLARY 2.1. *$\mathrm{GL}_2(\mathbf{C})$ is generated by w and the subgroups $Z(\mathbf{C}), A(\mathbf{C}), N(\mathbf{C})$. $\mathrm{SL}_2(\mathbf{C})$ is generated by w, $A(\mathbf{C})$ and $N(\mathbf{C})$.*

Notice that this decomposition is "algebraic": let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{C}) - N(\mathbf{C})A(\mathbf{C})$ $(c \neq 0)$, then $g = n(u)wn(v)a(w)$

$$g.\infty = a/c = n(u)0 = u,$$

hence $n(-a/c)g.z = g.z - a/c = \dfrac{az+b}{cz+d} - a/c = -\dfrac{1}{c(cz+d)} = wn(v)a(w) = \dfrac{-1}{w^2z+v}$

so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & cd \\ 0 & 1 \end{pmatrix} \begin{pmatrix} cd & 0 \\ & 1/cd \end{pmatrix},$$

In particular we have

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}$$

COROLLARY 2.2. *The above decompositions remain valid if one replace $\mathrm{GL}_2(\mathbf{C})$, $\mathrm{SL}_2(\mathbf{C})$, $B(\mathbf{C})$, $N(\mathbf{C}), \dots$ etc. by $\mathrm{GL}_2(K)$, $\mathrm{SL}_2(K)$, $B(K)$, $N(K), \dots$ for $K \subset \mathbf{C}$ any subfield. In particular these remain valid for $K = \mathbf{R}$.*

REMARK 2.1. This, of course, could have been checked directly; the point of this proof is that it generalize to much more complicated groups of matrices.

2.4. The action of $\mathrm{GL}_2(\mathrm{C})$ on lines.

DEFINITION 2.1. *A line in $\mathrm{P}^1(\mathrm{C})$ is either $L \cup \{\infty\}$ where L is a line in $\mathbf{C} \simeq \mathbf{R}^2$ or a circle in $\mathbf{C}$.*

PROPOSITION 2.2. *The fractional transformations preserve lines.*

PROOF. It is sufficient to verify this for elements of Z, N, A and for w. This is obvious for Z, N, A since these correspond to affine transformations in $\mathbf{R}^2$.Consider w: since $wA = Aw$ and the elements of A act on $\mathbf{C}$ by affine transformations, we may assume that the line is symmetric about the imaginary axis $i\mathbf{R}$ and passes through 0 or i (correspond either to a horizontal line through 0 or i) or a circle centered on $i\mathbf{R}$ passing through 0 or i. In the former case this is easy and one always obtain an horizontal line (using the fact that $\Im wz = \Im z/|z|^2$). In the later case, the circle is parametrized by

$$C(\theta) = \theta \to i - r(e^{i\theta} - i),\ r \in \mathbf{R}$$

and (since the line $i\mathbf{R} \cup \{\infty\}$ is transformed into it self by w and) one check that $wC(\theta)$ parametrize the circle centered on $i\mathbf{R}$ whose diameter is the segment with endpoints

$$wC(\pi/2) = i,\ wC(-\pi/2) = \frac{i}{1+2r}.$$

□

3. Fractional linear transformations for real matrices

We consider the restriction of this action to the subgroup $\mathrm{GL}_2(\mathbf{R})$.

3.1. Orbits. This action has two orbits: the real projective line

$$\mathrm{P}^1(\mathbf{R}) = \mathbf{R} \cup \{\infty\},\ \mathrm{P}^1(\mathbf{C}) - \mathrm{P}^1(\mathbf{R}) = \mathbf{H}^+ \cup \mathbf{H}^-$$

with

$$\mathbf{H}^\pm = \{z = x + iy \in \mathbf{C},\ \pm y > 0\}$$

the upper and lower half-planes.

More precisely, $\mathrm{GL}_2^+(\mathbf{R})$ (equivalently $\mathrm{SL}_2(\mathbf{R})$) acts transitively on $\mathrm{P}^1(\mathbf{R})$: $N(\mathbf{R})w.\infty = \mathbf{R}$. The stabilizer of ∞ is

$$\mathrm{SL}_2(\mathbf{R})_\infty = B(\mathbf{R}) \cap \mathrm{SL}_2(\mathbf{R}) = B^1(\mathbf{R}) = N(\mathbf{R})A(\mathbf{R})$$

and the stabilizer of any other element of $\mathrm{P}^1(\mathbf{R})$ is conjugate to $B^1(\mathbf{R})$.

The subgroup B^1 act also transitively on $\mathbf{H}$: given $z = x + iy \in \mathbf{H}$, let

$$n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$$

then

$$n(x)a(y)i = \begin{pmatrix} y^{1/2} & x/y^{1/2} \\ & y^{-1/2} \end{pmatrix} i = z.$$

We set

$$m(z) := n(x)a(y) = \begin{pmatrix} y^{1/2} & x/y^{1/2} \\ & y^{-1/2} \end{pmatrix} \in M$$

The stabilizer of i is

$$\mathrm{SL}_2(\mathbf{R})_i = \mathrm{SO}_2(\mathbf{R}) = \{\ k(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},\ \theta \in \mathbf{R}.\}$$

and the stabilizer of any $z \in \mathbf{H}$ is conjugate to $\mathrm{SO}_2(\mathbf{R})$.

Finally any matrix of negative determinant for instance $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ map $\mathbf{H}^\pm$ to $\mathbf{H}^\mp$.

In particular

$$\mathrm{P}^1(\mathbf{R}) = \mathrm{SL}_2(\mathbf{R}).\infty = \mathrm{SL}_2(\mathbf{R})/N(\mathbf{R})A(\mathbf{R}),$$
$$\mathbf{H} = \mathrm{SL}_2(\mathbf{R}).i \simeq \mathrm{SL}_2(\mathbf{R})/\mathrm{SL}_2(\mathbf{R})_i = \mathrm{SL}_2(\mathbf{R})/\mathrm{SO}_2(\mathbf{R}).$$

3.2. Fixed points. The various fractional linear transformations of $\mathrm{SL}_2(\mathbf{R})$ are parameterized according to the number of their fixed points in $\mathrm{P}^1(\mathbf{C})$ and in particular are invariant under conjugation in $\mathrm{SL}_2(\mathbf{R})$: $g \in \mathrm{SL}_2(\mathbf{R})$ is

- $\pm Id$: its fixed points are all or $\mathrm{P}^1(\mathbf{C})$.
- Parabolic: $|\mathrm{tr}(g)| = 2$ or g has one fixed point on $\mathrm{P}^1(\mathbf{R})$:

$$g = n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, x \in \mathbf{R}$$

- Hyperbolic: $|\mathrm{tr}(g)| > 2$ or g has two fixed point on $\mathrm{P}^1(\mathbf{R})$:

$$g = a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix},\ y \in \mathbf{R}_{>0}.$$

- Elliptic: $|\mathrm{tr}(g)| < 2$ or g has one fixed point in each $\mathbf{H}^\pm$:

$$g = k(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},\ \theta \in [-\pi, \pi].$$

3.3. Action on half-lines. As we have seen the elements of $\mathrm{SL}_2(\mathbf{R})$ transform lines of $\mathrm{P}^1(\mathbf{C})$ into lines. Let us see what happens to the restriction of these lines on the upper-half space $\mathbf{H}$.

- If C is a circle in $\mathbf{H}$, $g.C$ is a circle contained in $\mathbf{H}$.
- If C is an horizontal line or a circle in $\mathbf{H}$ tangent to $\mathbf{R}$ at x (which correspond to a line in $\mathrm{P}^1(\mathbf{C})$ intersecting $\mathrm{P}^1(\mathbf{R})$ in one point, ∞ or $x \in \mathbf{R}$) then $g.C$ is either the circle in $\mathbf{H}$ tangent at the point $g.x$ (if $g.x \neq \infty$) or an horizontal line.
- If L is the restriction to $\mathbf{H}$ of a line in $\mathrm{P}^1(\mathbf{C})$ meeting $\mathrm{P}^1(\mathbf{R})$ in two distinct points, x, x' (L is either a non-horizontal half-line or the intersection of a circle with $\mathbf{H}$) then $g.L$ is the restriction to $\mathbf{H}$ of a line in $\mathrm{P}^1(\mathbf{C})$ meeting $\mathrm{P}^1(\mathbf{R})$ in $g.x, g.x'$

3.4. Various decompositions. Let us first recall the

PROPOSITION 2.3 (Bruhat decomposition). *One has*

$$\mathrm{GL}_2(\mathbf{R}) = B(\mathbf{R}) \sqcup N(\mathbf{R})wB(\mathbf{R}),$$
$$\mathrm{SL}_2(\mathbf{R}) = N(\mathbf{R})A(\mathbf{R}) \sqcup N(\mathbf{R})wN(\mathbf{R})A(\mathbf{R}).$$

and this decomposition is unique (a matrix g not in $B(\mathbf{R})$ can be written in a unique way into to form $g = nwb$ with $n \in N$ and $b \in B$).

One has also the following important decomposition

PROPOSITION 2.4 (Iwasawa decomposition).

$$\mathrm{SL}_2(\mathbf{R}) = NAK$$

and this decomposition is unique.

PROOF. Given g consider $z = x + iy = g.i = n(x)a(y).i$ where $a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ so that $(n(x)a(y))^{-1}g \in \mathrm{SL}_2(\mathbf{R})_i = \mathrm{SO}_2(\mathbf{R})$. □

We also have the following useful decomposition which we will prove later (but which can be proven directly)

PROPOSITION 2.5 (Cartan or "polar" decomposition).

$$\mathrm{SL}_2(\mathbf{R}) = KAK.$$

3.5. The disk model. It is useful (for instance for the purpose of visualization) to identify $\mathbf{H}$ with an open subset of a bounded domain in $\mathbf{C}$: this is accomplished by mean of the Cayley transform: for any $z_0 \in \mathbf{H}$, let

$$g_{C,z_0} = \begin{pmatrix} 1 & -z_0 \\ 1 & -\overline{z}_0 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C})$$

the corresponding fractional linear transformation on $\mathrm{P}^1(\mathbf{C})$

$$z \to g_{C,z_0}.z = \frac{z - z_0}{z - \overline{z}_0}$$

maps z_0 to 0 and is an holomorphic homeomorphism between $\mathbf{H}$ to the open unit disk $D(0,1)$:

$$|g_{C,z_0}.z| = |z - z_0|/|z - \overline{z}_0| < 1 \text{ iff } y > 0.$$

Moreover g_{C,z_0} maps $\mathrm{P}^1(\mathbf{R})$ bijectively to the unit circle and maps ∞ to 1. One advantage of this identification is that all points on $\mathrm{P}^1(\mathbf{R})$ play the same role.

In particular, the conjugate subgroup $g_{C,z_0}\mathrm{SL}_2(\mathbf{R})g_{C,z_0}^{-1}$ acts on $D(0,1)$ and the stabilizer of 0 is the conjugate of the stabilizer of z_0 and is given by the elements of the form:

$$\theta \in \mathbf{R} \mapsto e(\theta).$$

In particular, the action is given by Euclidean rotations with center 0.

To check this we remark that

$$g_{C,z_0} = y_0^{-1/2} g_{C,i}(n(x_0)a(y_0))^{-1} = y_0^{-1/2} g_{C,i} \begin{pmatrix} y_0^{1/2} & x_0 y_0^{-1/2} \\ 0 & y_0^{-1/2} \end{pmatrix}^{-1}$$

so that it is sufficient to check this for $z_0 = i$.

4. Topological group actions

So far we have considered only the set theoretic action of $\mathrm{GL}_2(\mathbf{C})$ on $\mathrm{P}^1(\mathbf{C})$ or of $\mathrm{GL}_2^+(\mathbf{R})$. But these groups and the space they are acting on are equipped with a natural topology (and in fact more structures) and we will now take this aspect into account.

4.1. The projective line and the Riemann sphere. The space $\mathrm{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$ has a topology of compact locally compact separated space with $\mathbf{C}$ an open dense subset: this is the *one point-compactification or Alexandroff compactification* of $\mathbf{C}$. On $\mathbf{C}$ one take the usual topology and one takes as a basis of open neighborhoods of ∞ the complement in $\mathrm{P}^1(\mathbf{C})$ of compact subsets of $\mathbf{C}$. The resulting topological space, $\widehat{\mathbf{C}}$ is separated locally compact and compact.

EXERCISE 4.1.1. Verify this and prove that $\mathrm{P}^1(\mathbf{R})$ is a closed (hence compact) subset.

REMARK 4.1. $\widehat{\mathbf{C}}$ is in homeomorphic via the stereographic projection map to the 2-sphere S_2: this is the *Riemann sphere.*

4.2. Topological groups. Besides this, the group $\mathrm{GL}_2(\mathbf{C})$ itself carries a natural topology: $\mathrm{GL}_2(\mathbf{C})$ an open subset of the $\mathbf{C}$-vector space of 2×2-matrices, $M_2(\mathbf{C}) \simeq \mathbf{C}^4$ which is equipped with the norm

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \|m\| = \mathrm{tr}(g{}^t\overline{g})^{1/2} = (|a|^2+|b|^2+|c|^2+|d|^2)^{1/2}.$$

DEFINITION 2.2. *A topological group G is a group which is also a topological separated locally compact space such that the map*

$$\begin{array}{ccc} G\times G & \to & G \\ (g,g') & \to & g^{-1}g' \end{array}$$

is continuous. In particular the translations and inversion maps

$$g \to gg',\ g \to g^{-1}$$

are homeomorphism.

The group $\mathrm{GL}_2(\mathbf{C})$ is a topological group and the subgroups $N(\mathbf{C})$, $A(\mathbf{C})$, $\mathrm{GL}_2(\mathbf{R})$, $\mathrm{SL}_2(\mathbf{R})$, $\mathrm{SO}_2(\mathbf{R})$ etc... are closed topological subgroups.

Boserve that $M_2(\mathbf{Z}) \subset M_2(\mathbf{R})$ is a discrete subset (the induced topology on $M_2(\mathbf{Z})$ is the discrete topology). This implies that $\mathrm{SL}_2(\mathbf{Z}) = M_2(\mathbf{R}) \cap \mathrm{SL}_2(\mathbf{R}) < \mathrm{SL}_2(\mathbf{R})$ is a discrete subgroup .