Topics in Number Theory: Gauss sums

Given a Dirichlet character $\chi \mod N$, let

$$G(\chi) = \sum_{a=1}^{N} \chi(a)e^{2\pi ia/N}$$

be the associated Gauss sum.

1. We will prove a theorem about the absolute value of such a sum, in the case where $\chi$ is primitive. There are multiple ways to prove this; we will do so from the perspective of the Plancherel theorem.

Recall the Plancherel theorem for $\mathbb{R}$:
Let $f$ be a function in the Schwartz class, and recall the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{2\pi ix\xi}dx.$$ 

Then $\|f\|_2 = \|\hat{f}\|_2$.

(a) Let’s prove an analogous version for $\mathbb{F}_p$. Given a function $f : \mathbb{F}_p \to \mathbb{C}$, define its Fourier transform as

$$\hat{f}(a) = \sum_{t \in \mathbb{F}_p} f(t)e^{2\pi iat/p}.$$ 

Prove that $\|\hat{f}\|_2^2 = p\|f\|_2^2$.

(b) Now let’s use this version of Plancherel’s theorem to show that, for $\chi$ that is primitive mod $p$, the absolute value of the Gauss sum is $\sqrt{p}$.

Hint: Show that $\hat{\chi}(a) = \chi(a^{-1})\hat{\chi}(1)$.

2. Prove the following:
If $\chi, \chi'$ are Dirichlet characters mod $N, N'$ (respectively) such that $(N, N') = 1$, then

$$G(\chi\chi') = \chi(N')\chi'(N)G(\chi)G(\chi').$$
3. Given non-primitive \( \chi \) mod \( N \), let \( \chi_0 \) be the primitive Dirichlet character mod \( N_0 \) which induces \( \chi \). If \( N_0 \) and \( N/N_0 \) are coprime, show:

\[
G(\chi) = \mu(N/N_0)\chi_0(N/N_0)G(\chi_0).
\]

You may use the fact that

\[
\sum_{1 \leq b \leq M, (b, M) = 1} e(b/M) = \mu(M).
\]