Topics in Number Theory: Gauss sums

Given a Dirichlet character χ mod N, let

$$G(\chi) = \sum_{a=1}^{N} \chi(a) e^{2\pi i a/N}$$

be the associated Gauss sum.

1. We will prove a theorem about the absolute value of such a sum, in the case where χ is primitive. There are multiple ways to prove this; we will do so from the perspective of the Plancherel theorem.

Recall the Plancherel theorem for $\mathbb{R}$:
Let f be a function in the Schwartz class, and recall the Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x\xi} dx.$$

Then $||f||_2 = ||\hat{f}||_2$.

(a) Let's prove an analogous version for $\mathbb{F}_p$. Given a function $f : \mathbb{F}_p \to \mathbb{C}$, define its Fourier transform as

$$\hat{f}(a) = \sum_{t \in \mathbb{F}_p} f(t) e^{2\pi i a t/p}.$$

Prove that $||\hat{f}||_2^2 = p||f||_2^2$.

(b) Now let's use this version of Plancherel's theorem to show that, for χ that is primitive mod p, the absolute value of the Gauss sum is $\sqrt{p}$.

Hint: Show that $\hat{\chi}(a) = \chi(a^{-1})\hat{\chi}(1)$.

2. Prove the following:
If χ, χ' are Dirichlet characters mod N, N' (respectively) such that $(N, N') = 1$, then

$$G(\chi\chi') = \chi(N')\chi'(N)G(\chi)G(\chi').$$

3. Given non-primitive χ mod N, let χ_0 be the primitive Dirichlet character mod N_0 which induces χ. If N_0 and N/N_0 are coprime, show:

$$G(\chi) = \mu(N/N_0)\chi_0(N/N_0)G(\chi_0).$$

You may use the fact that

$$\sum_{1 \leq b \leq M, (b,M)=1} e(b/M) = \mu(M).$$