Harmonic homogeneous polynomials: part 1

First, it is important to review what we have done up to this point:

Question 1. We proved the functional equation for Dirichlet L-functions over the past few weeks. Summarise the strategy used in **no more** than 3 lines.

Question 2. List the tools and strategies you used for the Gauss sum exercises from last week.

Let A be the algebra of polynomials in n variables with real coefficients. We will say that a polynomial P is *homogeneous* if all its monomials are of the same degree d. Note that the zero polynomial is considered to be homogeneous of any degree.

(Credit is due to R.Richard for some of the content below.)

Question 3.
(a) Show that, for any positive integer d, the space of homogeneous polynomials of degree d (in n variables) is of finite dimension.

We will denote this space as A_d.

(b) Calculate this dimension.
(c) Is A the sum of the subspaces A_d? Is it a direct sum? What about a direct product of the A_d? What if one replaces A by the algebra B of formal power series in n variables?

The Laplacian is a differential operator expressed as

$$\Delta = \sum_{i=1}^{n} \partial_i^2$$

where $\partial_i f = \frac{\partial f}{\partial x_i}$ is the partial derivative of f on its ith variable. We say that a polynomial P is *harmonic* if $\Delta P = 0$.

Question 4.
(a) Show, for every $P \in A$, that $\Delta(P)$ is well-defined, and is a polynomial.

(b) Prove that if P is homogeneous, then so is $\Delta(P)$. What is the degree of $\Delta(P)$?
(c) Show that a polynomial is harmonic iff each of its homogeneous components is harmonic.

We introduce on each of the spaces A_d a scalar product, given by

$$(S,T) = \sum_{\alpha} \alpha! \cdot S_\alpha T_\alpha$$

where α describes the n-tuple $(d_1, \ldots, d_n)$, with $\alpha!$ is defined as $(d_1, \ldots, d_n)! = d_1! \ldots d_n!$ and where $P_\alpha := P_{(d_1,\ldots,d_n)}$ denotes the coefficient, in any polynomial P, of the monomial $x_\alpha := x_1^{d_1} \ldots x_n^{d_n}$.

Let

$$Q = \sum_{i=1}^{n} x_i^2$$

be the polynomial corresponding to the square of the euclidean norm on $\mathbb{R}^n$.

Question 5.
Show that the maps

$$S \mapsto S \cdot Q : A_d \to A_{d+2}$$

and

$$T \mapsto \Delta(T) : A_{d+2} \to A_d$$

are linear maps that are adjoint to each other (with respect to the scalar product above).
Hint: For adjointness, show that for homogeneous polynomials P, Q,

$$(P,Q) = P(\partial)Q(x_1, \ldots, x_n) \mid_{x_1,\ldots,x_n=0},$$

where, for $P = \sum_\alpha P_\alpha x_\alpha$ we have $P(\partial) := \sum_\alpha P_\alpha \partial_\alpha$, with $\partial_\alpha := \partial_1^{d_1} \ldots \partial_n^{d_n}$.

(To be continued next week.)