Harmonic homogeneous polynomials: part 2

We continue from last week, keeping the same definitions and notation.
Let $H_d = \mathrm{Ker}(\Delta \mid_{A_d})$.

Question 6.
Use what we have seen so far to 'naturally' decompose A_{d+2} into two non-trivial orthogonal subspaces.

Question 7.
(a) Prove that every homogeneous polynomial P admits a unique expression as a finite sum (we will call it *harmonic decomposition*)

$$P = \sum_{i \geq 0} Q^i P_i$$

where the P_i are harmonic polynomials, where only finitely many are non-zero.
(b) Find the degree of the P_i's in the decomposition. For a given degree d, what is the highest-indexed P_i that can be non-zero?
(c) Find the dimension of H_d.

Question 8.
(a) Describe harmonic decomposition when $n = 1$.
(b) Describe harmonic decomposition when $n = 2$. Hint: Set $z = x + iy$, where x, y are your two polynomial variables, and observe what can be said about monomials in z.
(c) What can you say about the harmonic properties of a polynomial f restricted to the unit sphere S^{n-1} in $\mathbb{R}^n$?

Question 9.
(a) Recall the full Stone-Weierstrass theorem.
(b) Prove that every continuous function on the sphere is a uniform limit of restrictions to the sphere of harmonic polynomial functions on $\mathbb{R}^n$.

Now let μ be the rotation invariant measure on S^{n-1} of volume one (the latter meaning that $\mu(S^{n-1}) = 1$). It will be convenient to have the following formula: Let $\mathbf{v}$ be a vector valued smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$. Then we have

$$\int_{\mathbf{x} \in S^{n-1}} \mathbf{v} \cdot \mathbf{x} \; d\mu(\mathbf{x}) = \int_{\mathbf{x} \in B_n} \mathrm{div}(\mathbf{v}) \; d\lambda(\mathbf{x}),$$

where $\mathbf{v} \cdot \mathbf{x}$ is the euclidean scalar product, $\mathrm{div}(\mathbf{v}) = \sum_{i=1}^{n} \partial_i v_i$ is the divergence (scalar) function attached to $\mathbf{v}$, and $d\lambda$ is the n-dimensional Lebesgue measure on the euclidean unit ball B_n. The idea between this formula (and actually its proof) is the following. The left hand side is what is called the flux of the vector field $\mathbf{v}$ through the sphere S^{n-1}, which measures how much do the vector field cross the sphere. The divergence function $\mathrm{div}(\mathbf{v})$ evaluated at a point measures the flux at an infinitesimal level around a small sphere around the considered point. The formula asserts that the global flux through the sphere can be obtained by summing up all the divergence contained in the enclosed ball.

Question 10.
(a) Find the mean of a homogeneous harmonic polynomial on the sphere (with respect to μ).
(b) Find the mean of a harmonic polynomial on the sphere (with respect to μ).

(To be continued further next week.)