Harmonic homogeneous polynomials: part 3

Same notations and definitions as last week. Please make sure you finish the questions from part 2 first.

Question 11.
Let f and g be two homogeneous harmonic polynomials of non-zero degrees p and q respectively. Use the flux divergence formula to determine the value of (f, g) when $p \neq q$.

Denote by ${L^2}_d$ the subspace of $L^2(S^{n-1}, \mu)$ generated by restrictions of homogeneous harmonic polynomials of degree d.

Question 12.
(a) Find a non-zero polynomial whose restriction to sphere is zero.
(b) Prove that the restriction to the sphere of a non-zero homogeneous polynomial is non-zero.
(c) Deduce that restriction to the sphere defines an injective map with dense image, from the space of harmonic polynomials to $L^2(S^{n-1}, \mu)$. (Hint: first prove that H_d maps bijectively onto ${L^2}_d$).

We now consider the special orthogonal group $\mathrm{SO}(n)$. It is made of isometries of the euclidean space $\mathbb{R}^n$ which are linear and of determinant 1.

Recall that a *linear representation* of a group G on a (possibly infinite dimensional) vector space V is a group morphism ρ from G to the group $GL(V)$ of linear automorphisms of V. One can describe this representation by specifying, for arbitrary g in G and v in V, the action of $\rho(g)$ on v.

We will write $g \cdot v$ instead of $\rho(g)(v)$, and $\mathbf{X}$ for the vector $(X_1, \ldots, X_n)$.

Question 13.
(a) Does the formula,

$$(P \cdot g)(\mathbf{X}) = P(g(\mathbf{X})),$$

for g in $\mathrm{SO}(n)$, define a representation of $\mathrm{SO}(n)$ on the space of polynomial functions P on $\mathbb{R}^n$?
(b) Find a non-trivial representation of $\mathrm{SO}(n)$.
(c) Does the action of G preserve the uniform norm? What about the L^2 norm?
(d) Show that if P is a homogeneous polynomial function of some degree d, then so is $P \cdot g$.

(e) Show that if P is a harmonic polynomial function, then so is $P \cdot g$. (Do this for $n = 2$ if you find the general case too hard.)

Question 14.
Let P be a polynomial function on $\mathbb{R}^n$ which is invariant under the action of $\mathrm{SO}(n)$. Prove that P may be written as a polynomial of Q (that is, that P belongs to $\mathbb{R}[Q]$), where Q is the squared euclidean norm.

(Continues next week.)