Harmonic homogeneous polynomials and theta functions

Question 15:

Let V be a finite dimensional $SO(n)$-stable subspace of $L^2(S^{n-1}, \mu)$. Consider an orthonormal basis $f_1, \dots, f_n$ of V, and define

$$\psi_V(x, y) = \sum_{i=1,\dots,n} f_i(x) \cdot f_i(y).$$

(i) Prove that $\psi_V(x, y)$ does not depend on the choice of an orthonormal basis.
(ii) Determine $\psi_V(g \cdot x, g \cdot y)$ for g in $SO(n)$.
(iii) Prove that $x \mapsto \psi_V(x, P)$, where P is the north pole of the sphere, is a G-invariant function, where $G = \mathrm{stab}_{SO(n)}(P)$.
(iv) Prove that the previous function belongs to V, and is non zero if and only if V is not $\{0\}$.

Question 16:

Let A be a positive-definite symmetric matrix, and let P be a spherical function with respect to A. Denote by B the matrix equal to $\sqrt{\Lambda}U$, where U is an orthogonal matrix that diagonalises A, and Λ is the corresponding diagonal matrix.
Define

$$\Theta(z) = \sum_{m \in \mathbb{Z}^r} P(m) e\left(\frac{1}{2} A[m] z\right)$$

where $z \in \mathbb{H}$ and $A[x] = \ ^T x A x$.

(i) Prove that this is holomorphic in the upper-half plane.

Now we assume that A is integral and that $A^* = NA^{-1}$ is also integral, where N is any suitable fixed integer.

(ii) What is a suitable value for N?
For any suitable N, show that every prime factor of $|A|$ is in N.

(iii) If the degree ν of the spherical function is odd, what can you say?

(iv) Under what integer translations is Θ invariant?
We say that A is *even* if its diagonal elements are even. What does this tell us about the associated quadratic form $A[x]$?
If A is even, under what integer translations is Θ invariant?

Question 17:
We now want to prove

Proposition 0.1.

$$\sum_{m\in\mathbb{Z}^r} P(m+x)e\left(\frac{1}{2}A[m+x]z\right) = \frac{i^{-\nu}}{\sqrt{|A|}}\left(\frac{i}{k}\right)^k \sum_{m\in\mathbb{Z}^r} P^*(m)e\left(\frac{-A^{-1}[m]}{2z} + {}^T mx\right). \tag{0.1}$$

where $P^*(x) = P(A^{-1}x)$ is a spherical function for A^{-1} and $k = \nu + r/2$.

(i) What does this tell us when $x = 0$?

To determine this, we will need Poisson summation:

$$\sum_m f(m+x) = \sum_m \hat{f}(m)e({}^T mx)$$

and also the Fourier integral

$$\int_{\mathbb{R}} e\left(\frac{1}{2}y^2 z - yu\right) dy = \left(\frac{i}{z}\right)^{1/2} e\left(\frac{-u^2}{2z}\right)$$

where $u \in \mathbb{R}$ and $z \in \mathbb{H}$.

(ii) Let $P = 1$, start with the left-hand side of (0.1) and use Poisson summation to obtain the right-hand side. You may also want to make the substitution (in an integral) of $y = Bx$ (where B is defined as above).

(iii) Now without loss of generality we may assume that $P(x) = ({}^T cAx)^\nu$ for a suitable vector c. Define the differential operator

$$L = \sum_j c_j \partial/\partial x_j.$$

Use (multiple) applications of this to what you proved in (ii) to obtain (0.1).

(iv) What does equation (0.1) simplify to when $x = 0$ and $|A| = 1$?