Theta functions (continued)

As with last week, we begin with:

Let A be a positive-definite symmetric matrix, and let P be a spherical function with respect to A.
Define

$$\Theta(z) = \sum_{m \in \mathbb{Z}^r} P(m) e\left(\frac{1}{2} A[m] z\right)$$

where $z \in \mathbb{H}$ and $A[x] = {}^T x A x$.

We will now need the following:

Proposition 0.1. *Let* $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *with d odd. If $c \equiv 0 \bmod N$, then*

$$\Theta(\tau z) = \vartheta(\tau)(cz+d)^k \Theta(z),$$

where

$$\vartheta(\tau) = \left(\frac{|A|}{d}\right)\left(\bar{\epsilon}_d \left(\frac{2c}{d}\right)\right)^r,$$

and ϵ_d is equal to 1 if $d \equiv 1 \bmod 4$ and i if $d \equiv -1 \bmod 4$.

We will also make use of:

Theorem 0.2. *let N be such that NA^{-1} is integral. Let the diagonal of A be even. Then $\Theta(z)$ is an automorphic form for $\Gamma_0(2N)$ of weight $k = \nu + r/2$ and multiplier ϑ given as above. If $\nu > 0$, then it is a cusp form.*

Now let A be a symmetric positive definite integral matrix of even rank r. Let N be a positive integer such that NA^{-1} is also an integral matrix. Let A and NA^{-1} have even diagonal entries. Let P be a spherical function with respect to A of even degree ν.

We wish to show that under these conditions, Θ has a modular relation with respect to $\Gamma_0(N)$, where the weight is $k = \nu + r/2$ and the multiplier is as above.

Let $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

Question 18:

(i) What can we conclude if d is odd?

Note that if d is odd and negative, we will extend the Legendre-Jacobi symbol $\left(\frac{c}{d}\right)$ by the rule

$$\left(\frac{c}{d}\right) = \frac{c}{|c|}\left(\frac{c}{-d}\right)$$

as long as $c \neq 0$, and $\left(\frac{0}{d}\right)$ is equal to 1 if $d = \pm 1$ and 0 otherwise.

(ii) What happens to the multiplier system ϑ when r is even?

(iii) Find $\vartheta(-1)$ as a function of the rank.

Assume from here on that d is even.

(iv) Show that N is odd.

(v) Use this to show that $|A|$ is odd. Then prove that

$$D := (-1)^{r/2}|A| \equiv 1 \bmod 4.$$

(vi) Determine that $\vartheta(d) = \vartheta(c + d)$.

(vii) Consider $\Theta(\tau' z')$ where $\tau' = \tau T$ (T being the unit translation matrix in $SL_2(\mathbb{Z})$) and $z' = z - 1$. Use this to find a modular relation for $\Theta(\tau z)$.

(viii) What can you say about the rank r if N is equal to 1?