CHAPTER 1

Theta functions

1. Riemann's zeta function

What is now called the Riemann's zeta function was in fact introduced much earlier by Euler to give an analytical proof of the infiniteness of the prime numbers: Euler considered the function of the real variable $s > 1$

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s}.$$

This function is continuous (in fact real analytic) for $\Re s > 1$ and the *fundamental theorem of arithmetic* (every non-zero integer factors in an essentially unique into a product of prime powers) is equivalent to the factorization of $\zeta(s)$ (as an Euler product)

$$\zeta(s) = \sum_{n \geqslant 1} \frac{1}{n^s} = \prod_{p \ prime} (\sum_{\alpha \geqslant 0} \frac{1}{p^{\alpha s}}) = \prod_{p} (1 - p^{-s})^{-1}.$$

If the set of primes were finite, the righthand side would give that $\zeta(s)$ would converge to the finite product $\prod_p (1 - p^{-1})^{-1}$ as $s \to 1^+$ which clearly contradict the fact (proven by comparing with an integral) that

$$\lim_{s \to 1^+} \sum_{n \geqslant 1} \frac{1}{n^s} = +\infty.$$

REMARK 1.1. In fact Euler proved more: the series of the inverses of the prime is divergent:

$$\sum_{p \ prime} \frac{1}{p} = +\infty.$$

Thus the distribution properties of the prime numbers are closely linked to the analytic properties of $\zeta(s)$ and the great idea of Riemann's 1859 memoir was to *complexify* the real variable s and to study the analytic properties of $\zeta(s)$ for s a complex number. This eventually lead him to formulate the famous Riemann's hypothesis and one essential ingredient of it is his proof of analytic continuation and function equation of $\zeta(s)$ over the whole complex plane.

Recall that the Euler Γ function is given by the integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

for[1] $\Re s > 0$ and is analytically continued to $\mathbf{C}$ (with simple poles at non-positive integers) via the functional equation $s\Gamma(s) = \Gamma(s+1)$.

[1] here we have isolated the measure $\frac{dt}{t}$ because this is the measure (unique up to scalars) on the group $(Rr_{>0}, \times)$ which is invariant under the group multiplications : the Haar measure

THEOREM 1.1 (Riemann). *The function*

$$\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

has an analytic continuation to $\mathbf{C}$ *with two simple poles at* $s = 0, 1$ *and satisfies the function equation*

$$\Lambda(s) = \Lambda(1-s).$$

PROOF. Riemann proved this result using the integral (Mellin transform) representation

$$\Lambda(s) = \int_0^\infty (\sum_{n\geqslant 1} e^{-\pi n^2 t}) t^{s/2}\frac{dt}{t} \tag{1.1}$$

valid for $\Re s > 1$. Indeed by a change of variable, one has for any $n \geqslant 1$

$$\pi^{-s/2}\Gamma(s/2)n^{-s} = \int_0^\infty e^{-t}(\frac{t}{n^2\pi})^{s/2}\frac{dt}{t} = \int_0^\infty e^{-\pi n^2 t}t^{s/2}\frac{dt}{t}$$

and summing the above identity over all n (for $\Re s > 1$ so that everything converges absolutely) on obtains (1.1).

Let

$$\omega(t) := \sum_{n\geqslant 1} e^{-\pi n^2 t}$$

Notice that the integral converges at ∞ for any s (since $\omega(t) \simeq e^{-\pi t}$, $t \to +\infty$) while it converges at 1 only for $\Re s > 1$ (since $\omega(t) \gg t^{-1/2}$, $t \to 0^+$). This lead us to split the integral into the ranges $]0, 1[$ and $[1, +\infty[$ and by a change of variable $t \leftrightarrow 1/t$ we obtain

$$\int_0^\infty e^{-\pi x^2}t^{s/2}\frac{dt}{t} = \int_1^\infty \omega(t)t^{s/2}\frac{dt}{t} + \int_1^\infty \omega(1/t)t^{-s/2}\frac{dt}{t}$$

Let

$$\theta(t) := \sum_{n\in\mathbf{Z}} e^{-\pi n^2 t} = 1 + 2\omega(t)$$

then

$$\theta(t) = \sum_{n\in\mathbf{Z}} f_t(n),\ f_t(x) := e^{-\pi t x^2}.$$

The advantage of passing to $\theta(t)$ is that the series is over the whole discrete subgroup $\mathbf{Z} < \mathbf{R}$ and one has the

THEOREM 1.2 (Poisson summation formula). *Let* $f \in \mathcal{S}(\mathbf{R})$ *be in the Schwartz class; one has for* $u \in \mathbf{R}$

$$\sum_{n\in\mathbf{Z}} f(n+u) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)e(-nu)$$

where

$$\widehat{f}:\ y \to \int_{\mathbf{R}} f(x)e(-xy)dx,\ e(x) := \exp(2\pi i x)$$

is the Fourier transform

Let us recall that the the normalized Gaussian is its own Fourier transform

$$\widehat{e^{-\pi x^2}}(y) = e^{-\pi y^2}$$

and by change of variable this implies that

$$\widehat{f}_t(y) = t^{-1/2}f_{1/t}(y) \tag{1.2}$$

and so by Poisson formula

$$\theta(1/t) = t^{1/2}\theta(t).$$

In particular letting $t \to +\infty$ we obtain

$$\lim_{\varepsilon \to 0^+} \varepsilon^{1/2}\theta(\varepsilon) = \lim_{t \to +\infty} \theta(t) = 1.$$

REMARK 1.2. We note for future use that if t is a complex number with $\Re t > 0$ the above formula remains valid (this follows by analytic continuation). Here $\sqrt{t}$ is interpreted as the branch of $t \to \exp(\frac{1}{2}\log t)$ on $\mathbf{C} - \mathbf{R}_{\leqslant 0}$ which takes value $\sqrt{t}$ for $t > 0$.

$$\int_0^\infty e^{-\pi x^2} t^{s/2} \frac{dt}{t} = \int_1^\infty \frac{\theta(t)-1}{2} t^{s/2} \frac{dt}{t} + \int_1^\infty \frac{t^{1/2}\theta(t)-1}{2} t^{-s/2} \frac{dt}{t}$$
$$= \int_1^\infty \frac{\theta(t)-1}{2} t^{s/2} \frac{dt}{t} + \int_1^\infty \frac{\theta(t)-1}{2} t^{(1-s)/2} \frac{dt}{t} + \frac{1}{s} + \frac{1}{1-s}$$

Now the first two integrals are holomorphic on $\mathbf{C}$ and the whole expression is invariant under $s \leftrightarrow 1-s$ □

2. The theta function on the upper-half plane

In order to say more about θ we complexity the variable t: write $it := z$ and set

$$\widetilde{\Theta}(z) = \theta(t) = \sum_{n \in \mathbf{Z}} \exp(\pi i n^2 z) = \sum_{n \in \mathbf{Z}} e(n^2 z/2)$$

then $z \to \widetilde{\Theta}(z)$ is rapidly converging series and defines an holomorphic function in the Poincare upper half-plane

$$\mathbf{H} = \{z \in \mathbf{C},\ \Im z > 0\}.$$

Moreover it follows from (4.2) and analytic continuation that the following functional equation holds:

$$\widetilde{\Theta}(-1/z) = \sqrt{-iz}\widetilde{\Theta}(z) \tag{2.1}$$

Another equation satisfied by $\widetilde{\Theta}(z)$ is 2-periodicity

$$\widetilde{\Theta}(z+2) = \widetilde{\Theta}(z).$$

These transformation may be interpreted as Moebius transformations on $\mathbf{H}$.

2.1. Interlude: fractional linear transformations.

Let

$$\mathrm{GL}_2^+(\mathbf{R}) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R}),\ \det(g) > 0\}$$

the group of 2×2-invertible matrices with positive determinant. That group acts on $\mathbf{H}$ via fractional linear transformations (or homography)

$$g.z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

and these are holomorphic transformations. Indeed one has

$$\Im(gz) = \det(g)\frac{\Im(z)}{|cz+d|^2} > 0$$

since $\det(g) > 0$. Moreover the group of scalar matrices have a trivial action:

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} z = z.$$

In the present case

$$z+2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.z = T^2 z,\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ -1/z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z = w.z$$

say. Hence, implicitly the function $z \to \widetilde{\Theta}(z)$ satisfies certains transformation when it is composed with $z \to \gamma.z$ for γ any element in the subgroup

$$\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle < \mathrm{SL}_2(\mathbf{Z}).$$

2.2. The transformation law for $\widetilde{\Theta}$. We will be more explicit and will describe how $\widetilde{\Theta}$ transforms under the action of the group of matrices

$$\Gamma_d(2) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ b, c \equiv 0(2)\} \subset \mathrm{SL}_2(\mathbf{Z})$$

(ie. the group of integral matrices of determinant 1 which are congruent to diagonal matrices modulo 2). This is indeed a (normal) subgroup of $\mathrm{SL}_2(\mathbf{Z})$: the kernel of the reduction modulo 2 map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} (\mathrm{mod}\, 2) \in \mathrm{SL}_2(\mathbf{Z}/2\mathbf{Z}).$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_d(2)$; if $c = 0$, then (recall $b = 2b'$ and $ad - bc = 1$)

$$\gamma = \pm \begin{pmatrix} 1 & 2b' \\ 0 & 1 \end{pmatrix} = \pm T^{2b'}$$

and

$$\widetilde{\Theta}(\gamma.z) = \widetilde{\Theta}(T^{2b'}.z) = \widetilde{\Theta}(z + 2b') = \widetilde{\Theta}(z).$$

Thus (up to changing γ to $-\gamma$) we may assume that $c > 0$. We have

$$\widetilde{\Theta}(\gamma z) = \widetilde{\Theta}(\frac{a}{c} - \frac{1}{c(cz+d)}) = \sum_n e(\frac{an^2}{2c}) \exp(-i\pi \frac{n^2}{c(cz+d)})$$

Observe that the first term depends only on the congruence class $n(\mathrm{mod}\, 2c)$ so we can write it as

$$\sum_{\alpha(2c)} e(\frac{a\alpha^2}{2c}) \sum_{n \in \mathbf{Z}} \exp(-\pi \frac{ic}{(cz+d)}(\frac{\alpha}{2c} + n)^2)$$

so that applying the Poisson summation formula for f_t with $t = \frac{ic}{(cz+d)}$ and $u = \alpha/2c$ we obtain

$$= \frac{1}{2}(\frac{cz+d}{ic})^{1/2} \sum_{n \in \mathbf{Z}} e(n^2 \frac{z}{2}) \sum_{\alpha(2c)} e(\frac{a\alpha^2 + 2n\alpha + dn^2}{2c}).$$

Observe that since c is even, 5

$$\sum_{\alpha(c)} e(\frac{a\alpha^2 + 2n\alpha + dn^2}{2c}) = \frac{1}{2} \sum_{\alpha(2c)} e(\frac{a\alpha^2 + 2n\alpha + dn^2}{2c})$$

We have $ad-bc=1$ so $ad\equiv 1(2c)$ (since b is even); let $\overline{a}$ be the multiplicative inverse of a in $(\mathbf{Z}/2c)^\times$, we have $d\equiv\overline{a}(2c)$

$$a\alpha^2+2n\alpha+dn^2\equiv a[\alpha^2+2\overline{a}n\alpha+(\overline{a}n)^2]=a(\alpha+\overline{a}n)^2(\mathrm{mod}\,2c)$$

so that by a change of variable $\alpha\leftrightarrow\alpha+\overline{a}n$ we obtain

$$\sum_{\alpha(2c)}e(\frac{a\alpha^2+n\alpha+dn^2}{2c})=\sum_{\alpha(2c)}e(\frac{a\alpha^2}{2c})=G(a;2c),$$

where

$$G(a;c)=\sum_{\alpha(c)}e(\frac{a\alpha^2}{c}).$$

Given (a,c) two coprime integers, the sum $G(a;c)$ is the so-called *Gauss sum* which will be evaluated later. For the moment, simply we observe that if b is coprime with c,

$$G(ab^2;c)=G(a;c). \tag{2.2}$$

This follows from the change of variable $\alpha\to b\alpha$. In particular if $ad\equiv 1(c)$

$$G(a;c)=G(ad^2;c)=G(d;c).$$

Oberve that these terms are independent of n so that (since $ad\equiv 1(2c)$)

$$\widetilde{\Theta}(\gamma z)=\frac{1}{2}G(a;2c)\frac{(cz+d)^{1/2}}{(ic)^{1/2}}\widetilde{\Theta}(z)=\frac{G(d;2c)}{2(ic)^{1/2}}(cz+d)^{1/2}\widetilde{\Theta}(z). \tag{2.3}$$

2.3. A second computation. We will now compute $\widetilde{\Theta}(\gamma z)$ in a different way: since $w^2=-Id$ we have

$$\widetilde{\Theta}(\gamma z)=\widetilde{\Theta}(\gamma.(-Id).z)=\widetilde{\Theta}(\gamma wZ),\ Z=w.z=-1/z.$$

Now

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}b&-a\\d&-c\end{pmatrix}=\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}$$

hence using (2.1) we are reduced to compute $\widetilde{\Theta}(\gamma' Z)$ for

$$\gamma'=\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\in \mathrm{SL}_2(\mathbf{Z}),\ a',d'\equiv 0(2).$$

Let assume also that $c'>0$ (ie. $d>0$). Writing again

$$\gamma' Z=\frac{a'}{c'}-\frac{1}{c'(c'Z+d')}$$

and performing a computation similar as before (using that a',d' are even and c' is odd) we find that

$$\widetilde{\Theta}(\gamma z)=G(a'/2;c')(\frac{c'Z+d'}{ic'})^{1/2}\widetilde{\Theta}(Z)=G(-c/2;d)(\frac{cz+d}{-diz})^{1/2}(-iz)^{1/2}\widetilde{\Theta}(z).$$

$$\widetilde{\Theta}(\gamma z)=\frac{G(-c/2;d)}{d^{1/2}}(cz+d)^{1/2}\widetilde{\Theta}(z). \tag{2.4}$$

Comparing (2.3) and (2.4), we obtain

THEOREM 1.3. *For $c, d > 0$ coprimes and, c even:*

$$\frac{G(-2c;d)}{d^{1/2}} = \frac{G(d;2c)}{2(ic)^{1/2}}. \tag{2.5}$$

REMARK 2.1. This relation is quite remarkable as it relates objects living in very "different world": $G(-2c;d)$ is associated to the ring $\mathbf{Z}/d$ (the c dependancy is only through its congruence class $c(\mod d)$) while $G(d;2c)$ is associated with the ring $\mathbf{Z}/2c$ (the d dependancy is only through its congruence class $d(\mod 2c)$). This is an instance of a *reciprocity law*.

3. The Jacobi symbol

We will use (2.5) to evaluate specific values of the Gauss sum which we recall is defined for $(a,c)=1$ by

$$G(a;c) := \sum_{\alpha(c)} e(\frac{a\alpha^2}{c}) \in \mathbf{C}.$$

Notice that $a \mapsto G(a;c)$ defines in fact a function on $(\mathbf{Z}/c)^\times$.

Observe first that

$$\overline{G(a;c)} = \sum_{\alpha(c)} e(-\frac{a\alpha^2}{c}) = G(-a;c).$$

Next for $c=2$ we have (for $a \equiv 1(2)$)

$$G(a;2) = G(1;2) = 1 + e(\frac{1}{2}) = 0.$$

For $c=4$ we obtain, computing the sum of four terms

$$G(d;4) = \sum_{a=1,\cdots,4} = \begin{cases} 2\sqrt{2}e^{i\pi/4} & d \equiv 1(4) \\ 2\sqrt{2}e^{-i\pi/4} & d \equiv 3(4) \end{cases}.$$

Therefore taking $c=2$ in (2.5) we obtain for d odd

$$G(-4;d) = G(-1;d) = d^{1/2} \times \begin{cases} 1 & d \equiv 1(4) \\ -i & d \equiv 3(4) \end{cases} \tag{3.1}$$

hence by complex conjugation $\overline{G(1;d)} = G(-1;d)$

$$G(4;d) = G(1;d) = d^{1/2} \times \begin{cases} 1 & d \equiv 1(4) \\ i = e(\frac{1}{4}) & d \equiv 3(4) \end{cases} = d^{1/2}\varepsilon_d \tag{3.2}$$

say.

REMARK 3.1. In particular

$$d \to \varepsilon_d^2 = (-1)^{\frac{d-1}{2}} = \chi_4(d)$$

is the unique non-trivial character of $(\mathbf{Z}/4)^\times$.

Next for $c=8$ we have

$$G(d;8) = 2(1 + e(\frac{d}{8}) + e(\frac{d}{2}) + e(\frac{d}{8})) = 4e(\frac{d}{8}) \tag{3.3}$$

and hence

$$G(-8;d) = d^{1/2}e(\frac{d-1}{8}),\ G(8;d) = d^{1/2}e(-\frac{d-1}{8}) = G(2;d).$$

3.1. The Jacobi symbol. For d odd, positive and $(c,d) = 1$ any integer, we define the Jacobi symbol as

$$(\frac{c}{d}) = \frac{G(c;d)}{G(1;d)}.$$

This is a function on $(\mathbf{Z}/d)^\times$; in fact, since

$$(\frac{c(c')^2}{d}) = (\frac{c}{d})$$

$(\frac{c}{d})$ defines a function on $(\mathbf{Z}/d)^\times/(\mathbf{Z}/d)^{\times 2}$ and we will compute this symbol through several reductions.

Observe that from the previous computations

$$(\frac{1}{d}) = 1,\ (\frac{-1}{d}) = (\overline{\varepsilon}_d/\varepsilon_d) = (\overline{\varepsilon}_d)^2 = \chi_4(d) = (-1)^{\frac{d-1}{2}},$$

$$(\frac{2}{d}) = e(-\frac{d-1}{8})\varepsilon_d^{-1} = \begin{cases} 1 & d \equiv 1,7(8) \\ -1 & d \equiv 3,5(8) \end{cases} = (-1)^{\frac{d^2-1}{8}} = \chi_8(d)$$

is a non-trivial Dirichlet character of modulus 8 (a group homomorphism $\chi : (\mathbf{Z}/8)^\times \to \mathbf{C}^\times$) which iis distinct from the character induced by χ_4 throught the reduction map

$$(\mathbf{Z}/8)^\times \mapsto (\mathbf{Z}/4)^\times.$$

Notice first that for $c > 0$ and even, one has for any d_1 coprime with c (using (2.5) and (2.2))

$$\frac{G(-2c;dd_1^2)}{(dd_1^2)^{1/2}} = \frac{G(dd_1^2;2c)}{2(ic)^{1/2}} = \frac{G(d;2c)}{2(ic)^{1/2}} = \frac{G(-2c;d)}{d^{1/2}}.$$

Now for any integer c coprime with dd_1 we can find a positive $c' \equiv 0(2)$ such that $c \equiv -2c'(dd_1^2)$ hence we have for any c coprime with dd_1

$$\frac{G(c;dd_1^2)}{(dd_1^2)^{1/2}} = \frac{G(c;d)}{d^{1/2}}.$$

In particular we have for $(c,dd_1) = 1$

$$(\frac{c}{dd_1^2}) = (\frac{c}{d}).$$

Thus we are reduced the problem of computing the Jacobi symbol to the case where d is an odd square-free number.

3.2. Application of the Chinese Remainder Theorem. Consider again for a moment the general case where d is not necessarily odd or squarefree. Suppose we have a factorization $d = d_1d_2$ with d_1 and d_2 coprimes; by the the Chinese Remainder Theorem, the map

$$(x_1,x_2) \in (\mathbf{Z}/d_1)^\times \times (\mathbf{Z}/d_2)^\times \mapsto x_1d_2 + x_2d_1 \in (\mathbf{Z}/d)^\times$$

is bijective. Using the congruence

$$(x_1d_2 + x_2d_1)^2 \equiv x_1^2d_2^2 + x_2^2d_1^2 (\mathrm{mod}\, d)$$

we obtain the factorization of the Gauss sum (recall d_1, d_2 are coprimes)

$$G(c;d) = G(d_2c;d_1)G(d_1c;d_2). \tag{3.4}$$

More generally if write

$$d = \prod_{p|d} p^{\alpha_p},\ d_p = d/p^{\alpha_p}$$

then $(p, d_p) = 1$ we obtain the factorization of the Gauss sum

$$G(c;d) = \prod_p G(d_pc;p^{\alpha_p}). \tag{3.5}$$

It is therefore sufficient to evaluate $G(c;d)$ when $d = p^\alpha$ is a prime power and by the previous argument, if p is odd it is sufficient to do it for $d = p$ a prime.

3.3. Gauss sums over primes and the Legendre symbol. Recall that the subgroup $(\mathbf{Z}/p)^{\times 2}$ is of index 2 in $(\mathbf{Z}/p)^\times$ so by (2.2), $G(c;p)$ can only take two values: $G(1;p)$ if c is a square modulo p or $G(b;p)$ for $b \in (\mathbf{Z}/p)^\times$ which is not a square modulo p. Moreover for such a b the maps

$$\alpha \to \alpha^2,\ \alpha \to b\alpha^2$$

are 2 to 1 maps from $(\mathbf{Z}/p)^\times$ to the set of quadratic residues (resp. quadratic non-residues) in $(\mathbf{Z}/p)^\times$. It follows that

$$G(1;p) + G(b;p) = 2 + 2\sum_{\alpha \in (\mathbf{Z}/p)^\times} e(\frac{\alpha}{p}) = 2\sum_{\alpha \in \mathbf{Z}/p} e(\frac{\alpha}{p}) = 0.$$

Thus $G(b;p) = -G(1;p)$ hence we see that for $(c,p) = 1$

$$(\frac{c}{p}) = \frac{G(c;p)}{G(1;p)} = \begin{cases} 1 & c \text{ is a quadratic residue modulo } p \\ -1 & c \text{ is not} \end{cases}.$$

Since the multiplication in $(\mathbf{Z}/p)^\times$ by a quadratic residue preserve the subsets of quadratic and non-quadratic residues while multiplication by a non-quadratic residue exchange the two sets we see that

$$(\frac{\cdot}{p}) : (\mathbf{Z}/p)^\times \to \{\pm 1\}$$

is a group homomorphism (a character of order 2): it satisfies

$$(\frac{cc'}{p}) = (\frac{c}{p})(\frac{c'}{p}).$$

This character is called the Legendre symbol. From this we deduce that

$$G(d_pc;p) = (\frac{c}{p})G(d_p;p)$$

and that taking products we obtain that, for any odd square-free $d > 0$

$$(\frac{c}{d}) = \frac{G(c;d)}{G(1;d)} = \prod_{p|d} \frac{G(d_pc;p)}{G(d_p;p)} = \prod_{p|d} (\frac{c}{p}).$$

We have proven the

THEOREM 1.4. *For $d=\prod_{p|d}p^{\alpha_p}$ an odd positive integer and c coprime with d one has*

$$(\frac{c}{d})=\prod_{p|d}(\frac{c}{p})^{\alpha_p}$$

where $c\to(\frac{c}{p})$ denote the Legendre symbol. In particular the Jacobi symbol $c\to(\frac{c}{d})$ is a character of $(\mathbf{Z}/d)^\times$ with value in $\{\pm1\}$:

$$(\frac{cc'}{d})=(\frac{c}{d})(\frac{c'}{d})$$

3.4. Reciprocity laws. Let us recall that using (2.5) we have established the following formulas for d odd

$$(\frac{1}{d})=1,\ (\frac{-1}{d})=\chi_4(d)=(-1)^{\frac{d-1}{2}}=\begin{cases}1 & d\equiv 1(4)\\ -1 & d\equiv 3(4)\end{cases}. \tag{3.6}$$

$$(\frac{2}{d})=\chi_8(d)=(-1)^{\frac{d^2-1}{8}}=\begin{cases}1 & d\equiv 1,7(8)\\ -1 & d\equiv 3,5(8)\end{cases} \tag{3.7}$$

Let c be positive and $(c,2d)=1$, we obtain from (2.5) and (3.5)

$$(\frac{-4c}{d})=\varepsilon_d^{-1}\frac{G(d;4c)}{2\sqrt{2}i^{1/2}c^{1/2}}=\varepsilon_d^{-1}\frac{G(cd;4)}{2\sqrt{2}i^{1/2}}\frac{G(4d;c)}{c^{1/2}}=\varepsilon_d^{-1}\varepsilon_{cd}^{-1}\varepsilon_c(\frac{d}{c})$$

since $(\frac{-4}{d})=\varepsilon_d^{-2}$ we obtain the following

THEOREM 1.5 (Reciprocity law for the Jacobi symbol). *Given $c,d>1$ two coprime odd integers, one has*

$$(\frac{d}{c})(\frac{c}{d})=\varepsilon_c\varepsilon_d\varepsilon_{cd}^{-1}=(-1)^{\frac{c-1}{2}\frac{d-1}{2}}=\chi_4(c)^{\frac{d-1}{2}}=\chi_4(d)^{\frac{c-1}{2}}. \tag{3.8}$$

REMARK 3.2. When $(c,d)=(p,q)$ are distinct primes (3.6), (3.7) and (3.8) become relations between Legendre symbols and give a way of determining whether an integer is quadratic residue modulo a prime:

$$(\frac{-1}{p})=(-1)^{\frac{p-1}{2}},\ (\frac{2}{p})=(-1)^{\frac{p^2-1}{8}},\ (\frac{p}{q})(\frac{q}{p})=(-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

these relations form the celebrated *Laws of quadratic reciprocity* discovered by C.-F. Gauss who provided a lot of very different proofs. These laws were later extended to the Jacobi symbol by Jacobi. The present proof (which is essentially due to Hecke) proceed differently and deduce this law from the transformation properties of the theta function and is analytic in essence.

3.5. Summary. The Jacobi symbol $(\frac{c}{d})$ has been defined on the set of pairs of coprime integers (c,d) with d positive and odd. To give a uniform presentation of the automorphy relation satisfied by $\widetilde{\Theta}(\gamma z)$ it is useful to define the *extended Jacobi symbol* on pairs of integers (c,d) with $d\equiv 1(\mathrm{mod}\,2)$ and taking values in $\{0,-1,1\}$ by:

(1) $(\frac{c}{d})=0$ if $(c,d)\neq 1$.
(2) For $c\neq 0$,

$$(\frac{c}{-d})=\frac{c}{|c|}(\frac{c}{d}),\ (\frac{0}{d})=\begin{cases}1 & \text{if } d=\pm1\\ 0 & \text{otherwise}\end{cases}. \tag{3.9}$$