Let us recall what we have proven so far:

THEOREM 1.6. *The extended Jacobi symbol has the following properties: let d denote an odd integer*

(1) *If $d = \prod_p p^{\alpha_p} > 0$, $c \to (\frac{c}{d})$ is a character of $(\mathbf{Z}/d)^\times$, more precisely*
$$(\frac{c}{d}) = \prod_{p|d} (\frac{c}{p})^{\alpha_p}$$
where for any odd prime p, $c \to (\frac{c}{p})$ denote the Legendre symbol modulo p.

(2) *For $c \neq 0$ map $d \to (\frac{c}{d})$ defines a character of $(\mathbf{Z}/4|c|)^\times$ which is even if $c > 0$ and odd for $c < 0$, ie.*
$$(\frac{c}{-d}) = (\frac{c}{d}) \text{ if } c > 0,\ (\frac{c}{-d}) = -(\frac{c}{d}) \text{ if } c < 0.$$

(3) *In particular*
$$(\frac{-1}{d}) = \chi_4(d) = (-1)^{\frac{d-1}{2}},\ (\frac{2}{d}) = \chi_8(d) = (-1)^{\frac{d^2-1}{8}},$$

(4) *and for c odd*
$$(\frac{c}{d}) = \chi_4(d)^{\frac{c-1}{2}}(\frac{d}{c}) \text{ if } c > 0,\ (\frac{c}{d}) = \chi_4(d)\chi_4(d)^{\frac{|c|-1}{2}}(\frac{d}{|c|}) \text{ if } c < 0$$

4. The automorphy relation

From these computation we obtain that $\widetilde{\Theta}(z)$ satisfies the following *automorphy relation*

THEOREM 1.7. *For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_d(2)$, one has*

$$\widetilde{\Theta}(\gamma.z) = (\frac{2c}{d})\varepsilon_d^{-1}(cz+d)^{1/2}\widetilde{\Theta}(z). \tag{4.1}$$

PROOF. Let $\gamma \in \Gamma_d(2)$. If $c = 0$, $d = \pm 1$ and $\gamma = \pm T^{2(b/2)}$: we have
$$\widetilde{\Theta}(\gamma.z) = \widetilde{\Theta}(z) = (\frac{0}{d})\varepsilon_d d^{1/2}$$
by the extension (3.9) of the Jacobi symbol. For $c \neq 0$ and $d > 0$, we have from (2.4) and the definition of the Jacobi symbol
$$\widetilde{\Theta}(\gamma.z) = (\frac{-c/2}{d})\frac{G(1;d)}{d^{1/2}}(cz+d)^{1/2}\widetilde{\Theta}(z) = (\frac{2c}{d})\chi_4(d)\varepsilon_d(cz+d)^{1/2}\widetilde{\Theta}(z)$$
$$= (\frac{2c}{d})\varepsilon_d^{-1}(cz+d)^{1/2}\widetilde{\Theta}(z)$$
since $\chi_4(d) = \varepsilon_d^2$ and $\varepsilon_d^4 = 1$. For $d < 0$ we replace γ by $-\gamma$ and obtain
$$\widetilde{\Theta}(\gamma.z) = (\frac{-2c}{-d})\varepsilon_{-d}^{-1}(-cz-d)^{1/2}\widetilde{\Theta}(z) = (\frac{2c}{d})\varepsilon_d^{-1}(cz+d)^{1/2}$$
by the properties of the extended Jacobi symbol. □

It is also helpful to define the following variant of the Riemann theta series
$$\Theta(z) := \widetilde{\Theta}(2z) = \widetilde{\Theta}(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} z) = \sum_{n\in\mathbf{Z}} e(nz^2).$$

We have for $\gamma \in \mathrm{GL}_2^+(\mathbf{R})$

$$\Theta(\gamma z) = \widetilde{\Theta}(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \gamma z) = \widetilde{\Theta}(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} 2z)$$

Now the conjugate subgroup

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_d(2) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \Gamma_0(4)$$

where

$$\Gamma_0(4) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ c \equiv 0(4)\}.$$

$\Gamma_0(4)$ is sometimes called the *Hecke-Iwahori* subgroup of level 4. From this we deduce that

COROLLARY 1.1. *For any* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, *one has*

$$\Theta(\gamma.z) = j_{1/2}(\gamma, z)^{1/2}\Theta(z) \tag{4.2}$$

where

$$j_{1/2}(\gamma, z) = (\frac{c}{d})\varepsilon_d^{-1} j(\gamma, z)^{1/2},$$
$$j(\gamma, z) = cz + d.$$

That sort of transformation law is typical of modular forms.

4.1. The cocycle relation. Since

$$\Theta(\gamma\gamma' z) = \Theta(\gamma(\gamma' z))$$

we obtain a *cocycle relation*: for $\gamma, \gamma' \in \Gamma_0(4)$ one has

$$j_{1/2}(\gamma\gamma', z) = j_{1/2}(\gamma, \gamma' z) j_{1/2}(\gamma', z). \tag{4.3}$$

Squaring it and noting that

$$(\frac{c}{d})^2 \varepsilon_d^2 = \chi_4(d) = \begin{cases} 1 & d \equiv 1(4) \\ -1 & d \equiv 3(4) \end{cases},$$

we obtain

$$\chi_4(dd') j(\gamma\gamma', z) = \chi_4(d) j(\gamma, \gamma' z) \chi_4(d') j(\gamma', z).$$

Hence

$$j(\gamma\gamma', z) = j(\gamma, \gamma' z) j(\gamma', z). \tag{4.4}$$

and in fact this cocycle relation is valid for $\gamma, \gamma' \in \mathrm{GL}_2^+(\mathbf{R})$.

REMARK 4.1. We see from the above computation that the map

$$\chi_4 : \begin{array}{ccc} \Gamma_0(4) & \mapsto & \{\pm 1\} \\ \gamma & \to & \chi_4(d) \end{array}$$

is in fact a group homomorphism (a character of $\Gamma_0(4)$):

$$\chi_4(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}) = \chi_4(cb' + dd') = \chi_4(dd') = \chi_4(d)\chi_4(d').$$

5. Theta series attached to harmonic polynomials

5.1. Higher powers. For $l \geqslant 1$ an integer, consider the function

$$z \mapsto \Theta_l(z) := \Theta(z)^l.$$

If satisfies for $\gamma \in \Gamma_0(4)$

$$\Theta_l(\gamma.z) = ((\frac{c}{d})\varepsilon_d^{-1})^l j(\gamma, z)^{l/2}\Theta_l(z). \tag{5.1}$$

In particular if l is even the automorphy relation simplify to

$$\Theta_l(\gamma.z) = \chi_4(\gamma)^{l/2} j(\gamma, z)^{l/2}\Theta_l(z) \tag{5.2}$$

where

$$\chi_4(\gamma) = (\frac{c}{d})^2\varepsilon_d^2 = \chi_4(d) = \begin{cases} 1 & d \equiv 1(4) \\ -1 & d \equiv 3(4) \end{cases},$$

and for $l \equiv 0(4)$ we obtain

$$\Theta_l(\gamma.z) = j(\gamma, z)^{l/2}\Theta_l(z) \tag{5.3}$$

These relations are typical of what will be called modular forms.

5.2. Theta series and functions on spheres. Using the original expression for Θ one sees easily that

$$\Theta_l(z) = \sum \sum_{n_1,\dots,n_l} \mathrm{e}((n_1^2 + \dots + n_l^2)z) = \sum_{n \geqslant 0} r_l(n)\exp(nz)$$

where

$$r_l(n) = |\{n_1^2 + \dots + n_l^2 = n\}|.$$

Thus $r_l(n)$ is the number of ways to write n as a sum of l squares of integers. A slightly different interpretation is to view $r_l(n)$ as the cardinality of the set of integral solutions of a diophantine equation namely,

$$\mathcal{R}_l(n) = \{\mathbf{x} = (x_1, \dots, x_l) \in \mathbf{Z}^l,\ Q_l(\mathbf{x}) = n\}$$

where

$$Q_l(x_1, \dots, x_l) = x_1^2 + \dots + x_l^2$$

is the Euclidean quadratic form; we say that $\mathcal{R}_l(n)$ is the set of *integral representations* of the integer n by the quadratic form Q_l. Expressed differently and more concretely $r_l(n)$ is also the number of vectors with integral coordinates which are on the sphere of radius $\sqrt{n}$.

5.2.1. *The size of* $\mathcal{R}_l(n)$. We first evaluate $r_l(n)$ for $l \geqslant 4$: the fact $r_l(n)$ has a generating series which is a modular form allows to give estimates $r_l(n)$. In particular, for $l = 4$ Jacobi proved the following beautiful formula: for $n \geqslant 1$

$$r_4(n) = 8(2 + (-1)^n)\sum_{d|n, 2\nmid d} d = 8(2 + (-1)^n)\prod_{p^\alpha || n, p>2} p^\alpha \frac{1 - 1/p^{\alpha+1}}{1 - 1/p}.$$

In particular $r_4(n) \geqslant 1$ for any positive integer n; in other terms, one has an "analytic proof of

THEOREM 1.8 (Lagrange four squares Theorem). *Every positive integer is a sum of four squares.*

REMARK 5.1. Observe that $r^*(n) = r_4(n)/8$ is multiplicative: for $(m,n) = 1$

$$r_4^*(mn) = r_4^*(m)r_4^*(n).$$

This is a manifestation of Lagrange's identity

$$(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2) =$$
$$()^2 + ()^2$$
$$+()^2 + ()^2.$$

We will admit this formula for the moment and use it to deduce an estimate for the number of representations of an integer as a sum of $l \geqslant 4$ squares:

PROPOSITION 1.1. *Suppose that either $l = 4$ and n is odd, or $l \geqslant 5$ then*

$$r_l(n) = n^{l/2-1+o(1)}, \textit{as } n \to +\infty.$$

PROOF. We consider the case n odd and $l = 4$, one has

$$r_4(n) = 8n \prod_{p^\alpha || n} \frac{1 - p^{-\alpha-1}}{1 - p^{-1}}.$$

We estimate the second factor:

$$\log(\prod_{p^\alpha || n} \frac{1 - p^{-\alpha-1}}{1 - p^{-1}}) \ll \sum_{p|n} \frac{1}{p} \leqslant \sum_{k \leqslant \omega(n)} \frac{1}{k}$$

where $\omega(n) = \sum_{p|n} 1$ the number of prime divisors of n. Since $2^{\omega(n)} \leqslant n$, one has $\omega(n) \leqslant \log(n)$ and

$$\log(\prod_{p^\alpha || n} \frac{1 - p^{-\alpha-1}}{1 - p^{-1}}) \ll \log(\log n)$$

hence

$$r_4(n) = n^{1+O(\frac{\log \log n}{\log n})} = n^{1+o(1)}.$$

Observe that for any n, one has $r_4(n) \leqslant n^{1+o(1)}$.

Consider now $l = 5$:

$$r_5(n) = \sum_{l \leqslant n^{1/2}} r_4(n - l^2) = \sum_{\substack{l \leqslant n^{1/2} \\ l \equiv n(2)}} r_4(n - l^2) + \sum_{\substack{l \leqslant n^{1/2} \\ l \not\equiv n(2)}} r_4(n - l^2).$$

The first term is non-negative and bounded by

$$\sum_{\substack{l \leqslant n^{1/2} \\ l \equiv n(2)}} r_4(n - l^2) \leqslant \sum_{\substack{l \leqslant n^{1/2} \\ l \equiv n(2)}} (n - l^2)^{1+o(1)} \leqslant n^{1+1/2+o(1)}.$$

The second term is evaluated similarly but with an upper and lower bound

$$\sum_{\substack{l \leqslant n^{1/2} \\ l \not\equiv n(2)}} r_4(n - l^2) = n \sum_{\substack{l \leqslant n^{1/2} \\ l \not\equiv n(2)}} (1 - \frac{l^2}{n})^{1+o(1)} = n^{1+1/2+o(1)}.$$

So

$$r_5(n) = n^{3/2+o(1)}$$

and the general case follows by recurrence:

$$r_l(n) = \sum_{l \leqslant n^{1/2}} r_{l-1}(n-l^2) = n^{(l-1)/2-1} \sum_{\substack{l \leqslant n^{1/2} \\ l \not\equiv n(2)}} (1-\frac{l^2}{n})^{(l-1)/2-1+o(1)} = n^{l/2-1+o(1)}$$

since

$$\sum_{\substack{l \leqslant n^{1/2} \\ l \not\equiv n(2)}} (1-\frac{l^2}{n})^{(l-1)/2-1+o(1)} = n^{1/2+o(1)}.$$

□

In particular, if $l \geqslant 5$ (or if $l = 4$ and n is odd) one has more and more points on S_l as $n \to \infty$.

5.3. Application to equidistribution. Any integral vector $\mathbf{x} \in \mathcal{R}_l(n)$ yields a point on S_l by projection

$$\mathbf{x} \mapsto \mathbf{x}/\|\mathbf{x}\| = \mathbf{x}/\sqrt{n} \in S_l$$

and one thing one would like to understand is how the set $\frac{1}{\sqrt{n}}\mathcal{R}_l(n) \subset S_l$ fill the unit sphere as n grows.

5.3.1. *Equidistribution.* The sphere S_l carries a unique probability measure μ_l which is invariant under the action of the orthogonal group $\mathrm{SO}_l(\mathbf{R})$: that measure is given for $\Omega \subset S_l$ a non-empty open subset by

$$\mu_l(\Omega) = \frac{\mu_{\mathbf{R}^l}(C(\Omega))}{\mu_{\mathbf{R}^l}(B_l(0,1))}$$

where $B_l(0,1) = \{\mathbf{x} \in \mathbf{R}^l,\ Q_l(\mathbf{x}) \leqslant 1\}$ is the unit ball and

$$C(\Omega) = \{\lambda\mathbf{x},\ \lambda \in [0,1],\ \mathbf{x} \in \Omega\}$$

is the solid angle supported by Ω.

We consider the average

$$\frac{1}{r_l(n)} \sum_{Q_l(\mathbf{x})=n} f(\frac{\mathbf{x}}{\sqrt{n}})$$

which is a sort of Riemann sums over the rescaled integral vectors of length $\sqrt{n}$.

THEOREM 1.9. *Given $l \geqslant 4$ then, as $n \to +\infty$ (and $n \equiv 1(2)$ if $l = 4$), one has, for any $f \in \mathcal{C}(S_l)$*

$$\frac{1}{r_l(n)} \sum_{Q_l(\mathbf{x})=n} f(\frac{\mathbf{x}}{\sqrt{n}}) \to \mu_l(f). \tag{5.4}$$

One then says that the sequence of sets

$$(\frac{1}{n^{1/2}}\mathcal{R}_l(n))_{n\geqslant 1}\ (n \equiv 1(2) \mathit{if}\ l = 4)$$

becomes equidistributed on S_l wrt μ_l. It follow by approximation that for any open subset $\emptyset \neq \Omega \subset S_l$

$$|\{\mathbf{x} \in \mathcal{R}_l(n),\ n^{-1/2}\mathbf{x} \in \Omega\}| \simeq \mu_l(\Omega)|\mathcal{R}_l(n)|,\ n \to +\infty.$$

We start with the following general

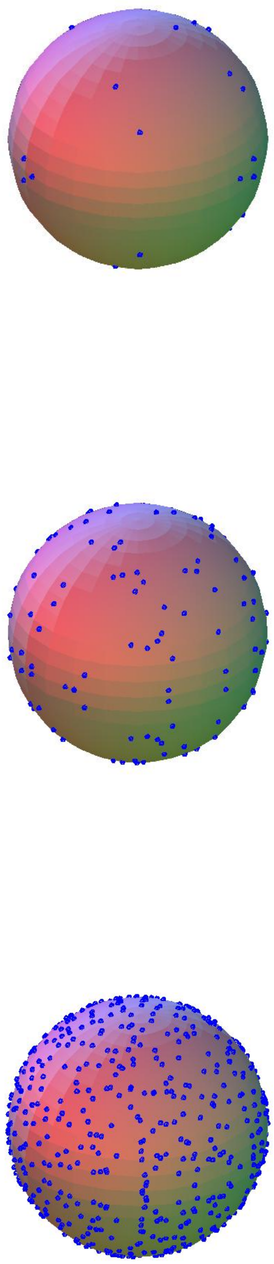

FIGURE 1. Distribution of $\mathcal{R}_4(5^\alpha)$, $\alpha = 2, 3, 4$ via the Hopf map $S_4 \to \mathrm{SO}_3(\mathbf{R})$

THEOREM 1.10 (Weyl's equidsitribution criterion). *In order for (5.4) to hold for any continuous function f it is sufficient that (5.4) holds for every $f \in \mathcal{F} \subset \mathcal{C}(S_l)$ for a family $\mathcal{F}$ generating a dense subspace in $\mathcal{C}(S_l)$ for the topology of uniform convergence.*

PROOF. Let f be a continuous function, by definition of $\mathcal{F}$ there exist for any $\varepsilon > 0$ a finite set I and a linear combination

$$f_I := \sum_{i \in I} \lambda_i f_i,\ \ \lambda_i \in \mathbf{C},\ \ f_i \in \mathcal{F}$$

such that

$$\|f - f_I\|_\infty \leqslant \varepsilon.$$

Therefore writing

$$\mu_{l,n}(f) = \frac{1}{r_l(n)} \sum_{Q_l(\mathbf{x})=n} f(\frac{\mathbf{x}}{\sqrt{n}})$$

we have

$$|\mu_{l,n}(f) - \mu_{l,n}(f_I)| \leqslant \mu_{l,n}(|f - f_I|) \leqslant \varepsilon.$$

Moreover for $n \geqslant n(\varepsilon)$ large enough we have

$$\mu_{l,n}(f_I) = \sum_i \lambda_i \mu_{l,n}(f_i) = \sum_i \lambda_i \mu_l(f_i) + O(\sum_i |\lambda_i|\varepsilon) = \mu_l(f_I) + O(\sum_i |\lambda_i|\varepsilon)$$

andso for n large enough we have

$$\mu_{l,n}(f) - \mu_{l,n}(f_I) \ll \varepsilon.$$

□

REMARK 5.2. Althougt this is discussed for the case of the sphere S_l only, Weyl's criterion is quite general and valid for any compact space and any sequence of probability measures: in order to show that a sequence of probability measures convergence weakly to some given measure, it is sufficient to test convergence against a family of functions generating a dense subspace of the space of continuous functions.

We need to produce such a family $\mathcal{F}$ in the case of the sphere:

DEFINITION 1.1. *A polynomial $P(\mathbf{x})$ on $\mathbf{R}^l$ is harmonic if it is homogeneous:*

$$P(\lambda \mathbf{x}) = \lambda^d P(\mathbf{x}),\ \ d = \deg P$$

and it is a zero eigenvalue of the Laplace operator:

$$\Delta_l P = 0,\ \ \Delta_l = \sum_{i=1\ldots l} \frac{\partial^2}{\partial^2 x_i}.$$

we denote by $\mathcal{H}_{l,d}$ the subspace of Harmonic polynomials of degree d.

The following follows from the exercise sessions:

THEOREM 1.11. *One has the following*

- *The group $\mathrm{SO}_l(\mathbf{R})$ acts on $\mathcal{H}_{l,d}$ (by linear change of variables $g.P(\mathbf{x}) = P(g^{-1}\mathbf{x})$) and $\mathcal{H}_{l,d}$ does not contain any proper $\mathrm{SO}_l(\mathbf{R})$-invariant subspace (in other terms $\mathcal{H}_{l,d}$ is an irreducible representation of $\mathrm{SO}_l(\mathbf{R})$).*

- $\mathcal{H}_{l,d}$ *is generated by poynomials of the shape*

$$P_{c,d}(\mathbf{x}) = (c.\mathbf{x})^d, c \in \mathbf{C}^l Q(c) = \sum_i c_i^2 = 0.$$

- *For* $d \neq d'$ *the subspaces* $\mathcal{H}_{l,d}$ *and* $\mathcal{H}_{l,d'}$ *are perpendicular with respect to the* $\mathrm{SO}_l(\mathbf{R})$*-invariant inner product on homogenenous polynomials*

$$\langle P, P' \rangle = \mu_l(P\overline{P}') = \frac{1}{vol(B_l)} \int_{B_l} P(\mathbf{x})\overline{P'}(\mathbf{x})d\mathbf{x}.$$

- *The subspace*

$$\sum_d \mathcal{H}_{l,d|S_l} \subset \mathcal{C}(S_l)$$

generated by the restriction to the sphere of the harmonic polynomials is dense in $\mathcal{C}(S_l)$ *for the topology of uniform convergence.*

The family $\bigoplus_{d\geqslant 0} \mathcal{H}_{d,l}$ generate a dense subspace of $\mathcal{C}(S_l)$ and because of Weyl's criterion, we may therefore assume that $f(\mathbf{x}) = P(\mathbf{x})$ for $P \in \mathcal{H}_{d,l}$ a non-constant harmonic polynomial: in that case, one has

$$\frac{1}{r_l(n)} \sum_{Q_l(\mathbf{x})=n} P(\frac{\mathbf{x}}{\sqrt{n}}) = \frac{1}{r_l(n)n^{d/2}} \sum_{Q_l(\mathbf{x})=n} P(\mathbf{x}).$$

We will show that for $l \geqslant 5$ and P non-constant,

$$r_l(n; P) := \sum_{Q_l(\mathbf{x})=n} P(\mathbf{x}) \ll n^{(l+d)/2-1-\delta}$$

for some $\delta > 0$. This implies that

$$\frac{1}{r_l(n)} \sum_{Q_l(\mathbf{x})=n} P(\frac{\mathbf{x}}{\sqrt{n}}) = \frac{1}{r_l(n)n^{d/2}} R_P(n) \ll n^{-\delta} \to 0 = \mu_l(P). \tag{5.5}$$

5.4. Theta series associated to harmonic polynomials. The proof of (5.5) Given an harmonic polynomial P one can form the theta series

$$\Theta(z; P) = \sum_{\mathbf{x}\in\mathbf{Z}^l} P(\mathbf{x})e(Q_l(\mathbf{x})z) = \sum_{n\geqslant 0} e(nz)(\sum_{Q_l(\mathbf{x})=n} P(\mathbf{x})) = \sum_{n\geqslant 0} r(n; P)e(nz)$$

say. Observe that if d is odd,

$$\Theta(z; P) = 0,$$

we may therefore assume that d is even. Since $|r(n; P)| \ll_P n^{l/2}$ this is a rapidly converging series hence an holomorphic function on $\mathbf{H}$. The following is a consequence of the Poisson summation formula:

THEOREM 1.12. *The theta series* $\Theta(z; P)$ *is an holomorphic function on* $\mathbf{H}$ *satisfying the following automorphy relations: for any* $\gamma \in \Gamma_0(4)$,

$$\Theta(\gamma z; P) = ((\frac{c}{d})\varepsilon_d^{-1})^l (cz + d)^{l/2+d} \Theta(z; P).$$

We will say that $\Theta(\gamma z; P)$ *is an holomorphic modular form of weight*

$$k = l/2 + d.$$