EXERCISE 4.2.1. Show that $\mathrm{SU}_2(\mathbf{C}) = \{g \in \mathrm{SL}_2(\mathbf{C}),\ g^t\overline{g} = \mathrm{Id}\}$ is a compact subgroup of $\mathrm{SL}_2(\mathbf{C})$ which is maximal for this property; show the same for $K = \mathrm{SO}_2(\mathbf{R}) < \mathrm{SL}_2(\mathbf{R})$.

EXERCISE 4.2.2. Show that the Cartan decomposition

$$N \times A \times K \to \mathrm{SL}_2(\mathbf{R})$$

is an homeomorphism (of course not a group morphism !).

DEFINITION 2.3. *A topological group action, $G \circlearrowleft X$, is an action of a topological group on a topological space such that the map*

$$\begin{array}{ccc} G \times X & \to & X \times X \\ (g,x) & \to (x, g.x) & \end{array}$$

is continuous.

EXERCISE 4.2.3. Verify that the action $\mathrm{GL}_2(\mathbf{C}) \times \mathrm{P}^1(\mathbf{C}) \to \mathrm{P}^1(\mathbf{C})$ is a topological group action.

5. Elements of hyperbolic geometry

The upper-half plane $\mathbf{H} = \mathbf{H}^+$, as a open subset of $\mathbf{C}$ is a complex variety equipped with two 1-forms

$$dz = dx + idy,\ d\overline{z} = dx - idy$$

and one has

$$d(gz) = \det g \frac{dz}{(cz+d)^2},\ \bar{d}(gz) = \det g \frac{d\overline{z}}{(c\overline{z}+d)^2}.$$

5.1. The hyperbolic metric. In particular (since $\Im(gz) = |cz+d|^{-2}\Im z$) the positive definite symmetric 2-forms

$$ds^2 = \frac{1}{y^2} dz \otimes d\overline{z} = \frac{dx \otimes dx + dy \otimes dy}{y^2}$$

is invariant under the action of $\mathrm{GL}_2^+(\mathbf{R})$:

$$\frac{1}{y(gz)^2} dgz \otimes d\overline{g}z = \frac{|cz+d|^4}{\det g^2 y^2} \frac{\det g^2 dz \otimes d\overline{z}}{(cz+d)^2(c\overline{z}+d)^2} = \frac{dz \otimes d\overline{z}}{y^2}.$$

In particular the corresponding Riemannian metric is $\mathrm{GL}_2^+(\mathbf{R})$-invariant; it is called the *hyperbolic metric.*

This give $\mathbf{H}$ the structure of a metric space for the geodesic distance by minimizing the length:

$$d_h(z,z') := \inf_L \ell_h(L),\ L \in \mathcal{C}^\infty([0,1] \to \mathbf{H}),\ L(0) = z,\ L(1) = w,$$

$$\ell_h(L) := \int_0^1 \frac{(x'(t)^2 + y'(t)^2)^{1/2}}{y(t)} dt,$$

and by invariance the group $\mathrm{GL}_2^+(\mathbf{R})$ act by isometries.

REMARK 5.1. One can in fact show that $\mathrm{PGL}_2(\mathbf{R})$ is the group of orientation preserving isometries of $\mathbf{H}$.

REMARK 5.2. The topology induced by this metric is the same as the one induced by the euclidean metric $dx^2 + dy^2$ (since both metrics are comparable when restricted to compact sets).

5.2. Shape of balls for the hyperbolic metric. We will now describe explicitly the hyperbolic circle and balls of hyperbolic radius r and center $z \in \mathbf{H}$.

$$S_h(z,r) = \{z' \in \mathbf{H},\ d_h(z,z') = r\},\ D_h(z,r) = \{z' \in \mathbf{H},\ d_h(z,z') \leqslant r\}.$$

The hyperbolic metric is invariant by $N(\mathbf{R})$ which is the group of horizontal translation on $\mathbf{H}$; so it is natural to look at the distance along a set of representatives of the space of N-orbits $N\backslash\mathbf{H}$, that is along a vertical half-line:

LEMMA 2.1. *Let $z = x+iy, z' = x+iy' \in \mathbf{H}$ with the same real part, then the geodesic segment between these two points is unique and is the vertical segment joining them and*

$$d_h(z,z') = |\int_{[y,y']} \frac{dt}{t}| = |\log y/y'|.$$

In particular for any $r \geqslant 0$, $d_h(z, x+iy\exp(\pm r)) = r$.

PROOF. Consider the vertical path joining z and z',

$$L_g(t) = x + i(ty' + (1-t)y) = x_g(t) + iy_g(t);$$

its length is given by

$$\ell_h(L) = \int_0^1 \frac{(x'(t)^2 + y'(t)^2)^{1/2}}{y(t)} dt = |\log(y'/y)|.$$

Conversely, given any other path joining z to z', since $\frac{(x'(t)^2+y'(t)^2)^{1/2}}{y(t)} \geqslant \frac{|y'(t)|}{y(t)}$, one has

$$\int_0^1 \frac{(x'(t)^2 + y'(t)^2)^{1/2}}{y(t)} dt \geqslant \int_0^1 \frac{|y'(t)|}{y(t)} dt \geqslant |\int_0^1 \frac{y'(t)}{y(t)} dt| = |\log(y'/y)|.$$

Moreover equality holds iff $x'(t) = 0\ \forall t$ that is iff $x(t)$ is constant equal to x. This prove that $d_h(z,z') = |\log y/y'|$ and the the vertical segment joining z to z' is the geodesic segment. □

PROPOSITION 2.6. *For $z, z' \in \mathbf{H}$, let $r = d_h(z,z')$, one has*

$$S_h(z,r) = \mathrm{SL}_2(\mathbf{R})_z.z'.$$

Moreover $S_h(z,r)$ is a (Euclidean) circle whose diameter is the segment with end-points

$$x + iy\exp(\pm r).$$

PROOF. Since $g \in \mathrm{SL}_2(\mathbf{R})_z$ is an isometry fixing z, one has

$$d_h(g.z', z) = d_h(g.z', g.z) = d_h(z,z') = r,\ ie.\ \mathrm{SL}_2(\mathbf{R})_z.z' \subset S_h(z,r).$$

For the converse: we observe that the stabilizer $\mathrm{SL}_2(\mathbf{R})_z$ is conjugate to $\mathrm{SL}_2(\mathbf{R})_i = \mathrm{SO}_2(\mathbf{R})$ so is compact. Thus the orbit $\mathrm{SL}_2(\mathbf{R})_z.z'$ is compact and let $z_\pm = x_\pm + iy_\pm \in \mathrm{SL}_2(\mathbf{R})_z.z'$ be such that y_+ (resp. y_-) is maximal (resp. minimal). We will show that

$$z_\pm = x + iy\exp(\pm r).$$

This imply that $\mathrm{SL}_2(\mathbf{R})_z$ acts transitively on $S_h(z,r)$ and that $S_h(z,r) = \mathrm{SL}_2(\mathbf{R})_z.z'$.

By definition we have $y_- \leqslant y\exp(-r)$; on the other hand, for $L : [0,1] \to \mathbf{H}$ any path between z and z_-, we have

$$\ell_h(L) = \int_0^1 \frac{(x'(t)^2 + y'(t)^2)^{1/2}}{y(t)} dt \geqslant \int_0^1 \frac{|y'(t)|}{y(t)} dt \geqslant |\log(y_-/y)| = \log(y/y_-).$$

Thus $r = \log(y/y_-)$; moreover $\ell_h(L) = \log(y/y_-)$ if only if $x'(t) \equiv 0$ so $x_- = x$ and $z_- = x + iy\exp(-r)$. Similarly, $z_+ = x + iy\exp(r)$.

Let us show that the hyperbolic circle is an euclidean circle: we may assume that $z = i$; indeed writing $z = g.i$ and $v = g.z'$, then $d_h(z, z') = d_h(i, v)$ and

$$\mathrm{SL}_2(\mathbf{R})_z.z' = g\mathrm{SL}_2(\mathbf{R})_i g^{-1}.gv = g\mathrm{SL}_2(\mathbf{R})_i.v$$

is the transform by g of the orbit $\mathrm{SL}_2(\mathbf{R})_i.v$ hence is a circle if $\mathrm{SL}_2(\mathbf{R})_i.v$ is. This later orbit is parametrized by $k(\theta)$:

$$k(\theta).v = \frac{\cos(\theta)v - \sin(\theta)}{\sin(\theta)v + \cos(\theta)} = \frac{e(\theta)(iv-1) + e(-\theta)(iv+1)}{e(\theta)(v+i) - e(-\theta)(v-i)} = g(v).e(2\theta),$$

$$g(v) = \begin{pmatrix} iv-1 & iv+1 \\ v+i & v-i \end{pmatrix} \in \mathrm{GL}_2(\mathbf{C}),$$

thus this orbit is the transform of the unit circle by the fractional linear transformation $g(v)$ hence is a circle. □

Corollary 2.3 (Cartan decomposition). *One has* $\mathrm{SL}_2(\mathbf{R}) = KAK$.

Proof. given $g \in \mathrm{SL}_2(\mathbf{R})$, let $z = g.i$ and let $k \in K$ such that $\Re(k.z) = \Re(i) = 0$, then there is $a \in A$ such that $k.z = a.i = kg.i$ so that $a^{-1}kg = k' \in K$. □

Observe that the Cartan decomposition is not unique. We have also the following:

Proposition 2.7. *The group* $\mathrm{SL}_2(\mathbf{R})$ *acts* 2*-transitively on* $\mathbf{H}$*: if* $d_h(z, z') = d_h(w, w')$ *there exists* $g \in \mathrm{SL}_2(\mathbf{R})$ *such that*

$$gz = w, \ \ gz' = w'.$$

Proof. Set $r = d_h(w, w')$. Take g such that $gz = w$, then $g.z' \in S_h(w, r)$, then there exist $k \in \mathrm{SL}_2(\mathbf{R})_w$ such that $kg.z' = w'$ and kg answer the question. □

From this we deduce

Proposition 2.8. *The geodesic segment joining two points* $z \neq z' \in \mathbf{H}$ *is unique and either the vertical segment between these two points if their real part agree or the arc of the unique half-circle centered on* $\mathbf{R}$ *containing these two points . Moreover, one has the formula*

$$\cosh(d_h(z, z')) = 1 + 2\frac{|z - z'|^2}{4yy'}.$$

Proof. By the previous proposition if $\Re z \neq \Re z'$ one can find g such that gz, gz' are vertically aligned and at the same distance. The geodesic segment joining z and z' is the transform by g^{-1} of the vertical segment between $[gz, gz']$. The verification of the formula for the hyperbolic distance is left to the reader. □

5.3. The hyperbolic metric in the disk model. Under the Cayley transform $z \to g_{C,z_0}.z = u = re^{i\theta}$ the hyperbolic metric transform into a metric proportional to the

$$\frac{du \otimes d\overline{u}}{(1 - |u|^2)^2}$$

This metric is radially invariant: invariant under euclidean rotations around 0, $u \mapsto e^{i\theta}u$. in particular the Cayley transform maps hyperbolic disks centered at z_0 to Euclidean disk centered at 0, geodesic segments passing through z_0 to Euclidean segment passing through 0.

5.4. The Hyperbolic measure. Similarly the alternating 2-form

$$\frac{1}{y^2}dz \wedge d\overline{z} = \frac{-2idx \wedge dy}{y^2}$$

is $\mathrm{GL}_2^+(\mathbf{R})$-invariant. Remove the $-2i$-factor, on obtainsf fmro this 2-form a measure on $\mathbf{H}$ of density

$$d\mu_h(z) = \frac{dxdy}{y^2}$$

called the *hyperbolic measure*. This measure is $\mathrm{GL}_2^+(\mathbf{R})$-invariant: for $f(z)$ continuous compactly supported and $g \in \mathrm{SL}_2(\mathbf{R})$

$$\int_{\mathbf{H}} f(z)d\mu_h(z) = \int_{\mathbf{H}} f(gz)d\mu_h(z).$$

This measure is called the *hyperbolic measure*.

CHAPTER 3

The action of $\mathrm{SL}_2(\mathbf{Z})$

We examine now the action of the subgroup $\mathrm{SL}_2(\mathbf{Z})$ in $\mathbf{H}$ and more generally the action of certain subgroups (congruence subgroups to be defined below) $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ of finite index. In particular we will explain how the structures on $\mathbf{H}$ (topological space, Riemannian manifold, complex manifold) descent to the space of orbits

$$Y(\Gamma) = \Gamma\backslash\mathbf{H} = \{\Gamma z,\ z \in \mathbf{H}\}.$$

We will show the following

THEOREM 3.1. *For $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ a congruence subgroup, the space $Y(\Gamma)$ has a structure of (non-compact) Riemann surface such that the projection map*

$$\pi_\Gamma : \mathbf{H} \mapsto Y(\Gamma)$$

is a local holomorphic homeomorphism at all but finitely many points (and at all points if $\Gamma = \Gamma(q),\ q \geqslant 3$). Moreover, there exist a Compact Riemann surface, $X(\Gamma)$ and an holomorphic embedding $Y(\Gamma) \hookrightarrow X(\Gamma)$ such that $X(\Gamma) - Y(\Gamma)$ is finite.

This will mainly come from the fact that action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathbf{H}$ is topological and that $\mathrm{SL}_2(\mathbf{Z})$ is a large discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$.

1. Congruence subgroups

DEFINITION 3.1. *For $q \geqslant 1$ an integer, the principal congruence subgroup of level q is the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of matrices congruent to the identity modulo q:*

$$\Gamma(q) := \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ a \equiv d \equiv 1(q),\ b \equiv c \equiv 0(q)\}.$$

$\Gamma(q)$ is is a finite index normal subgroup of $\mathrm{SL}_2(\mathbf{Z})$ and

$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma(q)] = |\,\mathrm{SL}_2(\mathbf{Z}/q)| = q^3 \prod_{p|q}(1 - 1/p^2).$$

EXERCISE 1.0.1. Prove the claim.

DEFINITION 3.2. *A subgroup $\Gamma < \mathrm{SL}_2(\mathbf{Z})$ is* congruence or arithmetic *if it contains some principal congruence subgroup. In particular it is of finite index in $\mathrm{SL}_2(\mathbf{Z})$*

Example of congruence subgroups are the Hecke-Iwahori subgroups

$$\Gamma_0(q) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod}\, q)\}.$$

Other important examples of arithmetic subgroups are

$$\Gamma_1(q) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\, q)\}$$

$$\Gamma_d(q) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} (\mathrm{mod}\, q)\}.$$

Note however that not all the finite index subgroups of $\mathrm{SL}_2(\mathbf{Z})$ are arithmetic.

Exercise 1.0.2. Compute the indexes of $\Gamma_0(q),\ \Gamma_1(q),\ \Gamma_d(q)$.

For specific congruence subgroups we use the following standard notations:

$$Y(q) := Y(\Gamma(q)),\ Y_1(q) := Y(\Gamma_1(q)),\ Y_0(q) := Y(\Gamma_0(q)).$$

2. The fundamental domains

For the maximal congruence subgroup $\mathrm{SL}_2(\mathbf{Z})$ we have the following

Theorem 3.2. *$\mathrm{SL}_2(\mathbf{Z})$ is generated by $n(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.*

Proof. Notice that $w^2 = -\mathrm{Id}$.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$. We proceed by recurrence on $|c|$: if $c = 0$ we are done since $\gamma = \pm n(b)$; if $c \neq 0$, we have for $k \in \mathbf{Z}$

$$n(k)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+kc & b+kd \\ c & d \end{pmatrix}.$$

so if $c \neq 0$ multiplying by a proper power of $n(1)$ we reduce to the case where $|a| < |c|$; next applying w we have

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & b-d \\ a & b \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

since $|c'| < |c|$ we conclude by recurrence. □

Definition 3.3. *Let $\Gamma < \mathrm{SL}_2(\mathbf{R})$ be a discrete subgroup. A fundamental domain for Γ, $\mathcal{D} \subset \mathbf{H}$ say, is an open subset whose closure meets every Γ-orbit in at least one point: ie. such that*

$$\forall z \in \mathbf{H},\ \Gamma.z \cap \overline{\mathcal{D}} \neq \emptyset,$$

and which meet every orbit in at most one point : i.e.

$$\forall z \in \mathbf{H},\ |\Gamma.z \cap \mathcal{D}| \leqslant 1.$$

Observe that the above conditions are equivalent to

$$\mathbf{H} = \Gamma.\overline{\mathcal{D}},\ \text{and}\ \forall \gamma \in \Gamma,\ \gamma \neq \pm Id,\ \mathcal{D} \cap \gamma.\mathcal{D} = \emptyset$$

Theorem 3.3. *A fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$ is given by the set*

$$\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})} = \{z \in \mathbf{H},\ |x| < 1/2,\ |z| > 1\}.$$

More generally for $\Gamma < \mathrm{SL}_2(\mathbf{Z})$ a subgroup of finite index:

$$\mathrm{SL}_2(\mathbf{Z}) = \bigsqcup_{\gamma_i} \Gamma\gamma_i$$

a fundamental domain for Γ is given by

$$\mathcal{D}_\Gamma = \bigcup_{\gamma_i} \gamma_i \mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}.$$

PROOF. Let $z \in \mathbf{H}$, we claim that there exist $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\Im(\gamma.z)$ is maximal: since $\Im(\gamma.z) = \Im(z)/|cz+d|^2$ this amount to find γ such that $|cz+d|$ is minimal. The map $(c,d) \in \mathbf{R}^2 \mapsto |cz+d|$ is a norm on $\mathbf{R}^2$ and since $\mathbf{Z}^2 \subset \mathbf{R}^2$ is discrete there exists $(c,d) \neq 0$ such that $|cz+d| > 0$ is minimal. Observe that the gcd $(c,d) = 1$: if $c = (c,d)c'$, $d = (c,d)d'$ then $|cz+d| = (c,d)|c'z+d'|$ contradicting minimality if $(c,d) > 1$. Given such (c,d), by Bezout's theorem, there exist $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ of the shape $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Replacing z by $\gamma.z$ we may assume that $\Im(z)$ is maximal within its orbit. In particular $\Im(w.z) = \Im(z)/|z|^2 \leqslant \Im z$ so $|z| \geqslant 1$ and applying $n(1)^k$ for a suitable $k \in \mathbf{Z}$, we have $n(1)^k.z = z+k$ which does not change the imaginary part, we may always assume that $\Re ez \in]-1/2, 1/2]$. This proves that for any $z \in \mathbf{H}$, $\mathrm{SL}_2(\mathbf{Z}).z \cap \overline{\mathcal{D}}_{\mathrm{SL}_2(\mathbf{Z})} \neq \emptyset$.

Suppose that $z \in \overline{\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}}$ we claim that for any coprime integers c, d,

$$|cz+d| \geqslant 1.$$

Indeed, if c or $d = 0$ (then d or $c = \pm 1$), the inequality is obvious; suppose $cd \neq 0$, we have

$$|cz+d|^2 = (cx+d)^2 + c^2y^2 = c^2(x^2+y^2) + 2cdx + d^2 \geqslant c^2 - |cd| + d^2 \geqslant 2|cd| - |cd| = |cd| \geqslant 1.$$

In particular for any $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ we have

$$\Im\gamma.z = \Im z/|cz+d|^2 \leqslant \Im z$$

so $\Im z$ is maximal within its orbit. Hence if z and $z' = \gamma.z \neq z$ are both in $\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}$ then $\Im(z) = \Im(\gamma.z)$ and $|cz+d| = 1$. If $c = 0$, $\gamma = \pm n(k)$ for $k \in \mathbf{Z}$ and $z' = z+k$. Hence $|k| = 1$, $\Re z = \pm 1/2$ and $\Re z' = \mp 1/2$. If $d = 0$ then $c = \pm 1$ and $|z| = 1$ so $z \in \partial\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}$. If $cd \neq 0$, then $|c| = |d| = 1$, $x = -cd/2 = \pm 1/2$, $|z| = 1$ and $y = \frac{\sqrt{3}}{2}i$ and $z = j$ or $-\overline{j}$ and n any case it belongs to $\partial\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}$

The final part of the statement is immediate from the equivalent definition of a fundamental domain. ☐

REMARK 2.1. From the above proof one see that under the $\mathrm{SL}_2(\mathbf{Z})$-action the boundary segments $[j, j+i\infty[$ and $[-\overline{j}, -\overline{j}+i\infty[$ are identified (by $n(\pm 1)$) and that the boundary segments $[i, j]$ is identified to $[i, -\overline{j}]$ via w. Moreover these are the only ways by which two distinct elements of the boundary of $\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}$ can be identified.

REMARK 2.2. The construction of this fundamental domain is obtained by considering points z with the greatest possible imaginary part within their orbit $\mathrm{SL}_2(\mathbf{Z}).z$. That is points in the orbit which are, in a certain sense, the "closest" to ∞. There are other similar (somewhat canonical) ways to produce nice fundamental domains in general; the following is due to Dirichlet: given $z_0 \in \mathbf{H}$ and $\Gamma < \mathrm{SL}_2(\mathbf{Z})$ a finite index subgroup then if $\Gamma_{z_0} \subset \{\pm \mathrm{Id}\}$

$$\overline{\mathcal{D}_{\Gamma,z_0}} = \{z \in \mathbf{H},\ d_h(z_0, z) = \min_{\gamma \in \Gamma - \{\pm \mathrm{Id}\}} d_h(z_0, \gamma.z)\}$$

is the closure of a fundamental domain.

3. The quotient topology

The reason why $\Gamma\backslash\mathbf{H}$ inherit many structures from $\mathbf{H}$ is that the action of $\mathrm{SL}_2(\mathbf{R})$ preserve these structures: the maps

$$\gamma. : z \in \mathbf{H} \mapsto \gamma.z$$

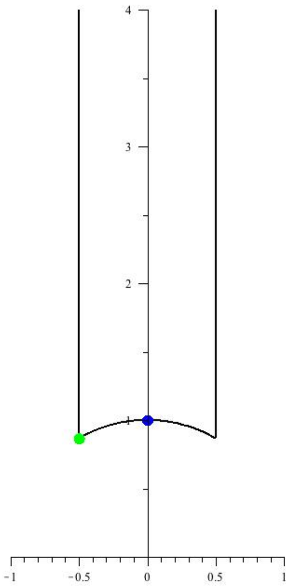

FIGURE 1. The fundamental domain for $\mathrm{SL}_2(\mathbf{Z})$ and the points i, j

are continuous, isometric (for the hyperbolic metric) and even holomorphic. The topological structure we'll put on $X(\Gamma)$ is the so-called quotient topology.

Let X be a locally compact separated space.

DEFINITION 3.4. *Let $G \circlearrowleft X$ be a group acting continuously on X, the quotient topology on the space of orbit $G\backslash X$ is the finest topology on $G\backslash X$ for which the projection map*

$$\pi_G : X \mapsto G\backslash X$$

is continuous.

Therefore a set on $G\backslash X$ is open if and only if its preimage under π_G is an open set. Moreover $\Omega \subset X$ is open

$$\pi_G^{-1}(\pi_g(\Omega)) = \bigcup_{g\in G} g.\Omega$$

is open so π_G is open. In particular the image under π_G of a basis of neighborhoods of a point x form a basis of neighborhoods of $G.x$. in particular, since the image under π_G of a compact is compact, $G\backslash X$ is locally compact.

3.1. Proper actions.

DEFINITION 3.5. *Let $G \circlearrowleft X$ be a topological group action on a topological space. G act properly on X if the (continuous)map*

$$\begin{array}{ccc} G\times X & \to & X\times X \\ (g,x) & \to (x, g.x) & \end{array}$$

is proper: the preimage of a compact is compact.

REMARK 3.1. If G act properly, any closed subgroup $\Gamma < G$ also acts properly. A compact group acts properly.

An important consequence is

PROPOSITION 3.1. *If $G \circlearrowleft X$ properly then $G\backslash X$ endowed with the quotient topology is separated.*

PROOF. Suppose that $Gx \neq Gx'$ we want to find open sets $x \in \Omega$, $x' \in \Omega'$ such that $G\Omega \cap G\Omega' = \emptyset$. Since $x \neq x'$ and X is locally compact, there exist precompact open neighborhoods K, K' of x, x' which are disjoint. Since the action is proper there is a compact subset $H \subset G$ such that

$$\forall g \in G - H,\ gK \cap K' = \emptyset.$$

Indeed the preimage of $K \times K'$ in $G \times X$ is compact and we take for H its projection to G. Since H is compact, Hx is compact and does not contain x' so we may take an open set $x' \in \Omega' \subset K'$ such that $Hx \cap \Omega' = \emptyset$ hence $x \notin H^{-1}\Omega'$. $H^{-1}\Omega'$ is precompact so we may choose a neighborhood $x \in \Omega \subset K$ such that $H.\Omega \cap \Omega' = \emptyset$. Then $G\Omega \cap K' = \emptyset$ and $G\Omega \cap G.K' = \emptyset$. □

PROPOSITION 3.2. $\mathrm{SL}_2(\mathbf{R})$ *acts properly on* $\mathbf{H}$.

PROOF. Let $\Omega \times \Omega' \subset \mathbf{H} \times \mathbf{H}$ be a product of compact subsets. We may assume that $\Omega' = \Omega$. The preimage is

$$\{(g,x) \in \mathrm{SL}_2(\mathbf{R}) \times \Omega,\ gx \subset \Omega\} \subset \{g \in \mathrm{SL}_2(\mathbf{R}),\ g\Omega \cap \Omega \neq \emptyset\} \times \Omega$$

but

$$\{g \in \mathrm{SL}_2(\mathbf{R}),\ g\Omega \cap \Omega \neq \emptyset\} \subset \{g \in \mathrm{SL}_2(\mathbf{R}),\ gm(\Omega)\mathrm{SO}_2(\mathbf{R}) \cap m(\Omega)\mathrm{SO}_2(\mathbf{R}) \neq \emptyset\}$$
$$= m(\Omega)\mathrm{SO}_2(\mathbf{R})(m(\Omega)\mathrm{SO}_2(\mathbf{R}))^{-1}$$

which is compact (here $m : z \in \mathbf{H} \mapsto m(z) \in B^1 < \mathrm{SL}_2(\mathbf{R})$). □

In general, has the following necessary condition for properness:

PROPOSITION 3.3. *If $G \circlearrowleft X$ is proper, for any pair of compact sets $K, K' \subset X$, the set*

$$\{g \in G, gK \cap K' \neq \emptyset\}$$

is compact. In particular for any $x \in X$ (taking $K = K' = \{x\}$)

$$G_x = \{g \in G, gx = x\}$$

is compact.

COROLLARY 3.1. $\mathrm{SL}_2(\mathbf{R})$ *does not act properly on* $\mathrm{P}^1(\mathbf{R})$*: the stabilizer of ∞ is not compact.*

EXERCISE 3.1.1. Show that the maps

$$n(x)a(y) \in NA \mapsto n(x)a(y)K \in \mathrm{SL}_2(\mathbf{R})/K,\ gK \in \mathrm{SL}_2(\mathbf{R})/K \mapsto g.i \in \mathbf{H}$$

are homeomorphisms.

3.2. Passing to a subgroup. Let $G' < G$ be a closed subgroup, we have a natural surjective projection map

$$\pi_{G'\backslash G} : \begin{array}{ccc} G'\backslash X & \mapsto & G\backslash X \\ G'x & \mapsto & Gx \end{array}.$$

It follows immediately from the definition of the quotient topology that this map is continuous; in particular $G'\backslash X$ is separated if $G\backslash X$ is.

4. Application to the modular group

Since $\mathrm{SL}_2(\mathbf{Z}) < \mathrm{SL}_2(\mathbf{R})$ is discrete hence closed one obtain:

COROLLARY 3.2. *The group* $\mathrm{SL}_2(\mathbf{Z})$ *and any of its subgroup* Γ *acts properly on* $\mathbf{H}$. *In particular* $Y(\Gamma) = \Gamma\backslash\mathbf{H}$ *equipped with the quotient topology is separated locally compact.*

Moreover, since a compact subset of a discrete group is finite, one obtain:

COROLLARY 3.3. *Let* $\Gamma < \mathrm{SL}_2(\mathbf{Z})$ *be a finite subgroup; For any* $z, z' \in \mathbf{H}$ *one has:*

(1) *for any balls* $r, r' \geqslant 0$, *the set*
$$\{\gamma \in \Gamma \text{ such that } \gamma D_h(z,r) \cap D_h(z',r') \neq \emptyset\}$$
is finite.
(2) *In particular, the stabilizer of* z, Γ_z *is finite.*
(3) *If* $z' \notin \Gamma.z$, *there exist* $r > 0$ *such that* $\Gamma.D_h(z,r) \cap D_h(z',r) = \emptyset$.
(4) *there exists* $r > 0$ *such that*
$$\{\gamma \in \Gamma \text{ s.t. } \gamma D_h(z,r) \cap D_h(z,r) \neq \emptyset\} = \Gamma_z.$$

Observe that since elements of Γ_z are hyperbolic rotations around z, one has for r small enough

$$\begin{aligned}(4.1)\quad \Gamma_z &= \{\gamma \in \Gamma \text{ s.t. } \gamma D_h(z,r) \cap D_h(z,r) \neq \emptyset\} \\ &= \{\gamma \in \Gamma \text{ such that } \gamma D_h(z,r) \cap D_h(z,r) = D_h(z,r)\}\end{aligned}$$

4.1. Stabilizer of $\mathrm{SL}_2(\mathbf{Z})$. From the previous discussion it is important to understand the shape of the stabilizers in $\mathrm{SL}_2(\mathbf{Z})$ and its congruence subgroups:

PROPOSITION 3.4. *For any* $z \in \mathbf{H}$, $\mathrm{SL}_2(\mathbf{Z})_z$ *is a finite cyclic group of order* $2, 4, 6$. *The second and third possibilities occur if and only if* $z \in \mathrm{SL}_2(\mathbf{Z}).i$ *or* $z \in \mathrm{SL}_2(\mathbf{Z}).j$, $j = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$.

PROOF. Observe that $\mathrm{SL}_2(\mathbf{Z})_z = \mathrm{SL}_2(\mathbf{Z}) \cap \mathrm{SL}_2(\mathbf{R})_z$ is the intersection of a discrete and a compact subgroup so is finite. Moreover $\mathrm{SL}_2(\mathbf{R})_z$ is conjugate to $\mathrm{SO}_2(\mathbf{R}) \simeq S_1$ so any finite subgroup of it is cyclic (given a finite subgroup $G < S_1$, let $z_0 = e^{i\theta_0} \in G$ with $\theta_0 \in [0, 2\pi]$ of minimal size, then z_0 generate G.) Given $z \in \mathbf{H}$; from the determination of the fundamental domain, we may assume that $|\Re z| \leqslant 1/2,\ |z| \geqslant 1$. Under this assumption, let us solve the equation
$$g.z = z, g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$$
where g is a generator of $\mathrm{SL}_2(\mathbf{Z})_z$. As we have seen, if $g \neq \pm\mathrm{Id}$, it has exactly two fixed points (z and $\overline{z}$) hence g is elliptic and $|\mathrm{tr}(g)| = |a + d| < 2$. Thus, either $\mathrm{tr}(g) = 0$ or $\mathrm{tr}(g) = \pm 1$. In the former case, g has minimal polynomial
$$X^2 + 1$$
and the subgroup generated by g is $\{\mathrm{Id}, g, -\mathrm{Id}, -g\}$. Up to replacing g by $-g$ we may assume that $c \geqslant 0$. One has $\det g = -a^2 - bc = 1$ so $c > 0$ and
$$cz^2 + (d-a)z - b = 0 = (cz)^2 - 2a(cz) - bc = Z^2 - 2aZ + 1 + a^2,$$
$$Z = cz = a \pm i,\ z = \frac{a \pm i}{c}.$$

Necessarily $|c| = 1$ (since $\Im z \geqslant \sqrt{3}/2$) and therefore $a = 0$ and $z = i$.

In the later case, up to changing g to $-g$ we may assume that $\mathrm{tr}(g) = 1$ and g has characteristic polynomial

$$X^2 - X + 1 = 0;$$

this is an irreducible polynomial adn therefore this is the minimal polynomial of g. Therefore g generate the subgroup

$$\{\mathrm{Id}, g, g^2 = g - \mathrm{Id}, g^3 = -\mathrm{Id}, g^4 = -g, g^5 = -g^2\}.$$

One has $\det g = 1 = a(1-a) - bc$ and

$$cz^2 + (1-2a)z - b = 0 = (cz)^2 + (1-2a)(cz) + 1 - a(1-a) = Z^2 + (1-2a)Z + 1 - a + a^2,$$

$$Z = cz = \frac{2a - 1 + \sqrt{(1-2a)^2 - 4(1-a+a^2)}}{2} = \frac{2a - 1 + \sqrt{-3}}{2},\ z = \frac{2a - 1 \pm i\sqrt{3}}{2c}.$$

Again $|c| = 1$ (since $\Im z \geqslant \sqrt{3}/2$) and since $|\Re z| = |a - 1/2| \leqslant 1/2$ and $a = 0$ or 1 so that $z = j$ or $j - 1 = n(-1)j$. □

If $\Gamma < \mathrm{SL}_2(\mathbf{Z})$ is a congruence subgroup $\Gamma_z < \mathrm{SL}_2(\mathbf{Z})_z$. In particular for the principal congruence subgroups we have

Corollary 3.4. *For $q \geqslant 3$, $\Gamma(q)_z = \{\mathrm{Id}\}$ for all $z \in \mathbf{H}$.*

Proof. For any $\gamma \in \Gamma(q)$, $\mathrm{tr}(\gamma) \equiv 2 \not\equiv 0, \pm 1 (q)$. □

5. Complex structure

The upper-half plane as an open subset of $\mathbf{C}$ has a natural complex structure (i.e. meromorphic functions on $\mathbf{H}$ are well defined); this structure indeed descent to $Y(\Gamma)$ (and in particular $Y(\Gamma)$ has a structure of differentiable variety). For this, is suffice to provide an holomorphic atlas for $Y(\Gamma)$: that is a collection

$$\{(U_i, \varphi_i)_{i \in I}\}$$

of local charts: $\{U_i\}_{i \in I}$ is an open covering of $Y(\Gamma)$ and

$$\varphi_i : U_i \mapsto \varphi_i(U_i) = V_i \subset \mathbf{C}$$

is an homeomorphism of U_i onto its image such that for $U_i \cap U_j \neq \emptyset$ the *transition* map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \mapsto \varphi_j(U_i \cap U_j)$$

is an holomorphic diffeomorphism. If $z \in U_i$, the variable

$$t_i := \varphi_i(z'),\ z' \in U_i$$

is called a local parameter (or local uniformizer) at z

5.1. Local parameters. It follows from Cor. 3.3 that for any $z \in \mathbf{H}$ there exists $r = r(z) > 0$ such that for any subgroup $\Gamma < \mathrm{SL}_2(\mathbf{Z})$, the projection map $\pi_\Gamma : \mathbf{H} \mapsto \Gamma\backslash\mathbf{H}$ induces an homeomorphism (recall the Γ_z is finite)

$$\Gamma_z\backslash\Gamma_z D_h(z, r) \simeq \pi_\Gamma(D_h(z, r));$$

actually Γ_z preserve the ball $D_h(z, r)$ so we have

$$\Gamma_z\backslash D_h(z, r) \simeq \pi_\Gamma(D_h(z, r)). \tag{5.1}$$

In particular when $\Gamma_z \subset \{\pm\mathrm{Id}\}$

$$D_h(z, r) \simeq \pi_\Gamma(D_h(z, r)).$$

In other terms $Y(\Gamma)$ is locally homeomorphic to $\mathbf{H}$ at any z such that $\Gamma_z \subset \{\pm\mathrm{Id}\}$. This is the case if $z \notin \mathrm{SL}_2(\mathbf{Z})\{i, j\}$ or for any z if $\Gamma = \Gamma(q)$ for $q \geqslant 3$ by Cor. 3.4.

To reveal the structure of $\Gamma_z \backslash D_h(z, r)$ we compose (5.1) with the Cayley transform at z, $g_{C,z}$: we obtain

$$\pi_\Gamma(D_h(z, r)) \simeq \Gamma_z \backslash D_h(z, r) \simeq \Gamma'_0 \backslash D(0, r')$$

for some $r' > 0$ and where Γ'_0 is a finite cyclic subgroup of complex rotations centered at 0: $\Gamma'_0 = e(\theta_z)^{\mathbf{Z}}$ with $\theta_z = 2\pi/l$, $l \in \{1, 2, 3\}$. The later quotient is homeomorphic to the disc $D(0, (r')^l)$ via the map

$$w \in D(0, r') \mapsto w^l \in D(0, (r')^l).$$

In other terms we have obtained for any $z \in \mathbf{H}$ and $r = r(z) > 0$ sufficiently small, a local homeomorphism

$$\varphi_{z,r} : U_{\Gamma.z,r} := \pi_\Gamma(D_h(z, r)) \simeq D(0, r''). \tag{5.2}$$

Such an homeomorphism will be called a local uniformizer for $Y(\Gamma)$ at $\Gamma.z$.

An holomorphic atlas is provided by the local uniformizers (5.2)

$$\{(U_{\Gamma.z,r_z}, \varphi_{z,r_z})\}_z$$

for z varying over a set of representatives of $Y(\Gamma)$. Therefore $Y(\Gamma)$ has a natural structure of non-compact Riemann surface. We explain below how to compactify it.

5.2. Compactification of $X(\Gamma)$. From the description of the fundamental domain we see that $Y(1)$ is not compact: a sequence of orbits or the shape $\mathrm{SL}_2(\mathbf{Z})z_n$ with $y_n \to +\infty$ has no converging subsequences. The shape of the fundamental domain suggest at least two possible compactifications : by adding a single point or by adding a circle. However two points in the fundamental domain with the same imaginary part y are at (hyperbolic) distance $\leqslant 1/y$ from each other so are becoming closer as y gets large; suggest to consider the one point compactification. We will denote it by

$$X(1) = Y(1) \cup \{\overline{\infty}\}$$

and describe now its topology.

By definition of the one point compactification, the neighborhoods of $\overline{\infty}$ are the complements in $X(1)$ of compact subsets of $Y(1)$. A basis of neighborhoods is obtained by taking the image in $Y(1)$ of the upper half-spaces $\mathbf{H}_Y = \{z \in \mathbf{H}, \Im z > Y\}$ for $Y > 0$: we note these neighborhoods

$$U_{\overline{\infty},Y} := \pi_{\mathrm{SL}_2(\mathbf{Z})}(\mathbf{H}_Y).$$

Given $z \in \mathbf{H}$ and $r > 0$, there exist $Y = Y(z, r) > 0$ such that

$$\forall \gamma \in \mathrm{SL}_2(\mathbf{Z}),\ \gamma\mathbf{H}_Y \cap D_h(z, r) = \emptyset;$$

indeed it suffice to take $Y > \sup_{z\in D(z,r)} \Im(z)$ as follows from the arguments leading to the determination of the fundamental domain of $\mathrm{SL}_2(\mathbf{Z})$. This implies that the resulting topology is separated so $X(\mathrm{SL}_2(\mathbf{Z}))$ is a separated compact space. Moreover (from the determination of the fundamental domain), the map

$$\pm n(\mathbf{Z}) \backslash \mathbf{H}_Y = \mathrm{SL}_2(\mathbf{Z})_\infty \backslash \mathbf{H}_Y \mapsto \mathrm{SL}_2(\mathbf{Z})_\infty \backslash \mathbf{H}_Y \simeq \pi_{\mathrm{SL}_2(\mathbf{Z})}(\mathbf{H}_Y)$$

is an homeomorphism for $Y > 1$ and the map

$$q_\infty : z \in \mathbf{H} \mapsto \exp(2\pi i z) \in D(0, 1),$$

induces an homeomorphism

$$\varphi_{\overline{\infty}} : U_{\overline{\infty},Y} \simeq D(0, \exp(-2\pi Y))$$

with

$$U_{\overline{\infty},Y} := \pi_{\mathrm{SL}_2(\mathbf{Z})}(\mathbf{H}_Y) \cup \{\overline{\infty}\},$$

(indeed as $\Im z \to +\infty$, $q_\infty(z) \to 0$). This is our the uniformizer at the point $\overline{\infty}$.

5.3. Compactification and cusps. This compactification is in fact compatible with the $\mathrm{SL}_2(\mathbf{Z})$-action and we use this to compactify more generally the $Y(\Gamma)$.

Observe that the orbit

$$\mathrm{SL}_2(\mathbf{Z}).\infty = \mathrm{P}^1(\mathbf{Q}) = \mathbf{Q} \cup \{\infty\}$$

the rational projective line : indeed if $a, c \in \mathbf{Z}$ are coprime, any matrix in $\mathrm{SL}_2(\mathbf{Z})$ of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (which exists by Bezout's theorem) map ∞ to a/c. Therefore for any finite index subgroup $\Gamma < \mathrm{SL}_2(\mathbf{Z})$, $\mathrm{P}^1(\mathbf{Q})$ decomposes into finitely many orbits which are called the *cusps* of Γ:

$$\mathrm{P}^1(\mathbf{Q}) = \bigsqcup_{i \in \mathrm{Cusp}(\Gamma)} \Gamma.x_i$$

where $x_i \in \mathrm{P}^1(\mathbf{Q})$ ranges over a set of representatives of $\mathrm{Cusp}(\Gamma)$

Let $\widehat{\mathbf{H}} := \mathbf{H} \cup \mathrm{P}^1(\mathbf{Q})$, we set

$$X(1) = \mathrm{SL}_2(\mathbf{Z})\backslash\widehat{\mathbf{H}} = \mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H} \sqcup \{\mathrm{SL}_2(\mathbf{Z})\backslash\mathrm{SL}_2(\mathbf{Z}).\infty\}$$

and more generally for $\Gamma < \mathrm{SL}_2(\mathbf{Z})$ with finite index

$$X(\Gamma) = \Gamma\backslash\mathbf{H} \cup \{\Gamma\backslash\mathrm{SL}_2(\mathbf{Z}).\infty\} = \Gamma\backslash\mathbf{H} \sqcup \mathrm{Cusp}(\Gamma)$$

A (separated) topology on $\widehat{\mathbf{H}}$ is given by defining a basis of neighborhoods of $x \in \mathrm{P}^1(\mathbf{Q})$, $U_{x,Y},\ Y > 0$ to be

$$U_{\infty,Y} := \{\infty\} \cup \mathbf{H}_Y, \quad \text{if } x = \infty$$

and if $x \neq \infty$

$$U_{x,Y} = \gamma.U_{\infty,Y}, \text{ for any } \gamma \in \mathrm{SL}_2(\mathbf{Z}) \text{ such that } \gamma.\infty = x;$$

(the $U_{x,Y}$ are disks in $\mathbf{H}$ tangent to $\mathbf{R}$ at x.)

as pointed out above, the group $\mathrm{SL}_2(\mathbf{Z})$ (and any of its congruence subgroup) does NOT act properly on $\widehat{\mathbf{H}}$: the stabilizer of any $x \in \mathrm{P}^1(\mathbf{Q})$ is conjugate to $\pm n(\mathbf{Z})$ and consequently is not finite: still we have the following extension of Cor. 3.3:

Proposition 3.5. *Let Γ be congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$, $x, y \in \mathrm{P}^1(\mathbf{Q})$ and $z \in \mathbf{H}$.*

- *For any $r, Y > 0$ the set*
$$\{\gamma \in \Gamma,\ \gamma U_{x,Y} \cap \mathcal{D}(z, r) \neq \emptyset\}$$
is finite for and empty if r and $1/Y$ are small enough.
- *If $y \notin \Gamma x$, then*
$$\{\gamma \in \Gamma,\ \gamma U_{x,Y} \cap U_{y,Y} \neq \emptyset\}$$
is finite for any $r > 0$ and empty if Y is large enough ($Y > 1$ suffice).
- *If Y is large enough ($Y > 1$ suffice),*
$$\{\gamma \in \Gamma,\ \gamma U_{x,Y} \cap U_{x,Y} \neq \emptyset\} = \Gamma_x.$$

PROOF. Exercise □

This proposition implies that $X(\Gamma)$ equipped with the quotient topology is a locally compact separated topological space which is moreover compact because Γ has finite index in $\mathrm{SL}_2(\mathbf{Z})$.

5.4. Local uniformizer at the cusps. Let $\overline{x} = \Gamma.x$ be a cusp. A local uniformizer at $\overline{x}$ is defined as follows: let $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ such that $\gamma.\infty = x$, for $Y > 1$, by the previous proposition, one has a local homeomorphism

$$U_{\overline{x},Y} := \pi_\Gamma(U_{x,Y}) \simeq \Gamma_x \backslash U_{x,Y} \simeq \gamma^{-1}\Gamma_x\gamma \backslash U_{\infty,Y}$$

We claim that the subgroup

$$\gamma^{-1}\Gamma_x\gamma \subset \mathrm{SL}_2(\mathbf{Z})_\infty = \pm n(\mathbf{Z})$$

has finite index in $\mathrm{SL}_2(\mathbf{Z})_\infty$: indeed $\gamma^{-1}\Gamma\gamma$ is a congruence subgroup (because if Γ contain the normal subgroup $\Gamma(q)$ so does $\gamma^{-1}\Gamma\gamma$, and $\Gamma(q)_\infty$ is of index q in $\mathrm{SL}_2(\mathbf{Z})_\infty$. Therefore $\gamma^{-1}\Gamma_x\gamma$ is of the form

$$n(h)^{\mathbf{Z}} \textit{ or } \{\pm \mathrm{Id}\}n(h)^{\mathbf{Z}}$$

for a unique $h \in \mathbf{N}_{>0}$. Then the map

$$q_x : z \in \mathbf{H} \mapsto q_\infty(\frac{1}{h}\gamma^{-1}.z) = \exp(2\pi i \frac{1}{h}\gamma^{-1}.z) \in D(0,1)$$

induces an homeomorphism

$$\varphi_{\overline{x}} : U_{\overline{x},Y} \simeq D(0, \exp(-2\pi Y/h))$$

and defines our local uniformizer at $\overline{x}$.

Indeed if suffice to check that the map q_x is Γ_x-invariant: for any $\gamma' \in \Gamma_x$,

$$\gamma^{-1}\gamma'\gamma = \pm n(hk),\ k \in \mathbf{Z}$$

and

$$q_x(\gamma'.z) = \exp(2\pi i \frac{1}{h} \pm n(hk)\gamma^{-1}z) = \exp(2\pi i \frac{1}{h}(\gamma^{-1}z + hk)) = q_x(z)$$

DEFINITION 3.6. *The integer h depend only on the cusp $\overline{x}$ and is called the width of $\overline{x}$.*

EXERCISE 5.4.1. Show that a set of representative $\mathrm{Cusp}(\Gamma_0(q))$, is given by the fractions

$$\frac{u}{v},\ v|q,\ 0 < u \leqslant (v, q/v).$$

Compute their width.

We conclude this section with the following:

THEOREM 3.4. *The altas $\{(U_{\Gamma.z,r_z}, \varphi_{z,r_z})\}_{z\in Y(\Gamma)} \cup \{(U_{\overline{x},Y}, \varphi_{\overline{x}})\}_{\overline{x}\in \mathrm{Cusp}(\Gamma)}$ is an holomorphic atlas gives $X(\Gamma)$ the structure of a compact Riemann surface. If $\Gamma' < \Gamma$, the natural projection map*

$$\pi_{\Gamma',\Gamma} : X(\Gamma') \mapsto X(\Gamma)$$

is a morphism of Riemann surfaces.

6. The hyperbolic measure

We have see that **H** carries two natural 2-forms, the hyperbolic metric

$$\frac{dz \otimes d\overline{z}}{y(z)^2}$$

and the hyperbolic measure

$$\frac{dz \wedge d\overline{z}}{y(z)^2},$$

which are both $\mathrm{SL}_2(\mathbf{R})$ invariant hence $\mathrm{SL}_2(\mathbf{Z})$-invariant. In particular these forms descend to corresponding forms on the quotients $Y(\Gamma)$. On the other hand these forms have singularities at the cusps:

Consider the cusp $\overline{c}$ with $\gamma c = \infty$; an uniformizer is given by (h the width)

$$z \to q_h = \exp(2\pi i \gamma z/h) \in D(0,1).$$

One has

$$dq_h = (2\pi i/h) q_h d(\gamma z),\ d\overline{q}_h = (-2\pi i/h)\overline{q}_h d\overline{\gamma z}$$

so that the hyperbolic metric and hyperbolic measure are given in these coordinates by

$$(2\pi/h)^2 \frac{dq_h \otimes d\overline{q}_h}{(|q_h| \log(1/|q_h|)/2\pi)^2},\ (2\pi/h)^2 \frac{dq_h \wedge d\overline{q}_h}{(|q_h| \log(1/|q_h|)/2\pi)^2}$$

which are singular as $q_h \to 0$.

6.1. The hyperbolic measure. The hyperbolic measure yields a corresponding hyperbolic measure on $Y(\Gamma)$; abusing notations, we denote it by

$$d\mu_\Gamma(\Gamma.z) = \frac{dxdy}{y^2},$$

and for f a μ_Γ-integrable function on $Y(\Gamma)$ we write

$$\mu_\Gamma(f) = \int_{Y(\Gamma)} f(\Gamma.z) d\mu_\Gamma(\Gamma.z) = \int_{Y(\Gamma)} f(\Gamma.z)\frac{dxdy}{y^2}.$$

A bit more concretely: the functions on $Y(\Gamma)$ are canonically identified with the functions on **H** which are Γ-invariant, through the map

$$f \in \mathcal{F}(Y(\Gamma)) \mapsto \tilde{f} \in \mathcal{F}(\mathbf{H})^\Gamma,\ \tilde{f}(z) := f(\Gamma.z),$$

and locally μ_Γ-integrable function on $Y(\Gamma)$ correspond to locally μ_h-integrable, Γ-invariant functions on **H**. For such functions one has

$$\mu_\Gamma(f) = \int_{\mathcal{D}_\Gamma} \tilde{f}(z) d\mu_h(z).$$

Notice that continuous bounded functions in $Y(\Gamma)$ are integrable: consider for instance the constant function 1 on $Y(\Gamma)$

$$\mu_\Gamma(1) = \int_{\mathcal{D}_\Gamma} d\mu(z) = \sum_{\gamma_i} \int_{\gamma_i \mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}} d\mu(z) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma] \int_{\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}} d\mu(z).$$

Now $\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})} \subset \{x + iy \in \mathbf{C},\ (x,y) \in [-1/2, 1/2] \times [\sqrt{3}/2, +\infty[\}$ and

$$\int_{\mathcal{D}_{\mathrm{SL}_2(\mathbf{Z})}} d\mu(z) \leqslant \int_{[-1/2,1/2]} \int_{[\sqrt{3}/2,\infty)} \frac{dxdy}{y^2} < \infty.$$

We define $\mathrm{vol}(Y(\Gamma)) = \mu_\Gamma(1)$ a simple computation shows that $\mathrm{vol}(Y(1)) = \frac{\pi}{3}$ so that

$$\mathrm{vol}(Y(\Gamma)) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma]\mathrm{vol}(Y(1)) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma]\frac{\pi}{3}.$$

Remark 6.1. Let us see again that bounded functions are integrable near the cusps: consider the disc coordinates

$$q_h = re(\theta),\ r = |q_h|, dq_h d\overline{q}_h = 4\pi r dr d\theta$$

and the hyperbolic measure becomes proportional to

$$\frac{dr d\theta}{r \log(1/r)^2},$$

and bounded functions near 0 are locally integrable against that measure.

6.2. The normalized hyperbolic measure. If $\Gamma' \subset \Gamma$, the space of integrable functions on $Y(\Gamma)$ inject naturally into the space of corresponding functions on $Y(\Gamma')$ via the obvious surjection $Y(\Gamma') \mapsto Y(\Gamma)$: in simple terms if f is Γ-invariant it is also Γ'-invariant. We reasoning we have done earlier for the constant function 1 shows that for any such function f

$$\mu_{\Gamma'}(f) = [\Gamma : \Gamma']\mu_\Gamma(f).$$

Therefore this lead us to define a *normalized measure* on the space of bounded functions on $\mathbf{H}$ which are invariant by some congruence subgroup by setting: for f Γ-invariant

$$\mu_n(f) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma]^{-1}\mu_\Gamma(f).$$

This definition does not depend on the choice of the congruence subgroup by which f is invariant: if f is Γ and Γ'-invariant then it is $\Gamma'' = \Gamma \cap \Gamma'$ invariant and since

$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma''] = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma][\Gamma : \Gamma''],\ [\mathrm{SL}_2(\mathbf{Z}) : \Gamma''] = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma'][\Gamma' : \Gamma'']$$

one has

$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma]^{-1}\mu_\Gamma(f) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma'']^{-1}\mu''_\Gamma(f) = [\mathrm{SL}_2(\mathbf{Z}) : \Gamma']^{-1}\mu'_\Gamma(f).$$