CHAPTER 3

The ring of Adèles

1. The strong approximation theorem

Let $v \in \mathcal{V}_{\mathbb{Q}} = \mathcal{V}$ be a place of $\mathbb{Q}$; we have seen that the inclusion

$$\delta_v : \mathbb{Q} \hookrightarrow \mathbb{Q}_v$$

has dense image. One may now consider this question for several places simultaneously: let $S \subset \mathcal{V}_{\mathbb{Q}}$ be a set and let $\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v$; the field $\mathbb{Q}$ embeds into $\mathbb{Q}_S$ via the diagonal embedding

$$\delta_S : x \in \mathbb{Q} \hookrightarrow (x, \cdots, x) \in \mathbb{Q}_S.$$

Theorem 3.1 (The weak approximation theorem). *For any finite set $S \subset \mathcal{V}_{\mathbb{Q}}$, the image of $\mathbb{Q}$ under the diagonal embedding is dense for the product topology on $\mathbb{Q}_S$.*

This theorem states that given any S-uple $(x_v)_{v \in S}$ one can find a rational number $x \in \mathbb{Q}$ such that x is *simultaneously* close to each x_v in the v-adic sense.

Proof. We may assume that $\infty \subset S$. It suffice to show that for any $r > 0$ and any uple $(x_v)_{v \in S} \in \mathbb{Q}_S$

$$\mathbb{Q} \cap \prod_v \Omega_v \neq \emptyset, \text{ where } \Omega_v = B_c(x_v, p_v^{-r})_v, \ p_\infty = e, \ p_p = p.$$

Actually we may assume that $x_v \in \mathbb{Q}$ for every v (by density of $\mathbb{Q} \subset \mathbb{Q}_v$). Shifting by $x_\infty \in \mathbb{Q}$ and replacing x_p by $x_p - x_\infty$ for $p \in S - \{\infty\}$, it is sufficient to show that for any $(x_p)_{p \in S - \{\infty\}} \in \prod_{p \in S} \mathbb{Q}$ and any integer $r > 0$

$$\mathbb{Q} \cap B_c(0, e^{-r})_\infty \prod_{p \in S} B_c(x_p, p^{-r})_p \neq \emptyset.$$

We start with the simpler problem of showing that

$$\mathbb{Q} \cap \prod_{p \in S} B_c(x_p, p^{-r})_p \neq \emptyset.$$

Multiplying by the product of the denominators of the x_p and changing r, if necessary, it suffice to show that for any $(x_p)_{p \in S} \in \prod_{p \in S} \mathbb{Z}$ and any $r > 0$

$$\mathbb{Z} \cap \prod_{p \in S} B_c(x_p, p^{-r})_p \neq \emptyset.$$

but this is equivalent to finding a solution to the system of congruences

$$x \equiv x_p \pmod{p^r} \text{ for every } p \in S.$$

By the Chinese reminder theorem such a solution always exists and can be choosen in the interval $[0, \prod_{p\in S} p^r]$. Let $q \not\subset S$ be another prime. By the above reasonning the system of congruences

$$x \equiv q^k x_p \,(\mathrm{mod}\, p^r) \text{ for every } p \in S$$

admits an integral solution;let $x_{q^k} \subset [0, \prod_{p\in S} p^r]$ be such a solution: we have

$$\frac{x_{q^k}}{q^k} \in \prod_{p\in S} B_c(x_p, p^{-r})_p$$

and in addition

$$|\frac{x_{q^k}}{q^k}|_\infty \leqslant \frac{\prod_{p\in S} p^r}{q^k} < e^{-r}$$

if k is taken sufficiently large.

□

1.1. The strong approximation theorem. In fact the proof given above yields a somewhat stronger statement:

THEOREM 3.2 (The strong approximation theorem). *Let $S \subset \mathcal{V}$ be a finite set of places of $\mathbb{Q}$, $v_0 \notin S$, and for each $v \in S$ let $\Omega_v \subset \mathbb{Q}_v$ be a non-empty open set. There exists $x \in \mathbb{Q}$ such that*

$$x \in \Omega_v \ \forall v \in S, \ \textit{and } x \in \mathbb{Z}_v \ \textit{ for all } v \notin S \cup \{v_0\}.$$

In particular if $\infty \notin S$ one may choose $v_0 = \infty$.

PROOF. Exercise. □

This theorem is stronger than the preceeding one because it states that, given any finite set of places S, any S-uple $(x_v)_{v\in S}$, one can always find a rational number $x \in \mathbb{Q}$ such that x is *simultaneously*

- close to each x_v in the v-adic sense, for each place $v \in S$
- a v-adic integer at all places v not in S, with at most one exception (which can be choosen at desired).

Our goal will be to reformulate this theorem in a more uniform way, in which the set of places S is less apparent. For this we consider the full infinite product rings

$$\mathbb{Q}_\mathcal{V} = \prod_{v\in\mathcal{V}} \mathbb{Q}_v = \mathbb{R} \times \prod_p \mathbb{Q}_p = \mathbb{Q}_\infty \times \mathbb{Q}_{\mathcal{V}_f}.$$

The field of rational numbers $\mathbb{Q}$ imbeds as a subring of $\mathbb{Q}_\mathcal{V}$ via the diagonal embedding

$$\delta_\mathcal{V} : x \in \mathbb{Q} \mapsto \delta(x) = (x_\infty = x, x_2 = x, x_3 = x, \cdots, x_p = x, \cdots) \in \mathbb{R} \times \prod_p \mathbb{Q}_p.$$

The strong approximation theorem in that case can be reformulated as follows:

THEOREM. *For any place v let $\Omega_v \subset \mathbb{Q}_v$ be a non-empty open subset and $\Omega = \prod_v \Omega_v \subset \mathbb{Q}_\mathcal{V}$. We assume that for all but finitely many primes p, $\Omega_p = \mathbb{Z}_p$, and also that for at least one $v \in \mathcal{V}$, $\Omega_v = \mathbb{Q}_v$, then*

$$\delta_\mathcal{V}(\mathbb{Q}) \cap \Omega \neq \emptyset.$$

2. The ring of Adèles

We would like to interpret this result as a sort of density result for $\delta_{\mathcal{V}}(\mathbb{Q})$:

(1) that would be possible if we declared the subsets $\prod_v \Omega_v$ with $\Omega_p = \mathbb{Z}_p$ for a.e. v a basis of open neighborhoods of $\prod_v \mathbb{Q}_v$ but this is not possible since there exist elements of $\prod_v \mathbb{Q}_v$ which are not contained in any of these sets (take for instance $(0, x_p)_p,\ x_p = p^{-p}$).

(2) We could instead equip $\prod_v \mathbb{Q}_v$ with the *product* or *Tychonoff* topology: the topology for which a basis of open sets is given by set of the shape $\prod_v \Omega_v$ with $\Omega_v \subset \mathbb{Q}_v$ non-empty open and $\Omega_v = \mathbb{Q}_v$ for a.e. v but then the density of $\delta_{\mathcal{V}}(\mathbb{Q})$ for this topology would be equivalent only to the weak approximation theorem and not to the strong one.

We observe that $\delta(\mathbb{Q})$ is contained in a significantly smaller subring of $\mathbb{Q}_{\mathcal{V}}$, namely

$$\mathbb{A} := \mathbb{R} \times \prod_p{}' \mathbb{Q}_p \text{ where } \prod_p{}' \mathbb{Q}_p = \{(x_p)_p,\ x_p \in \mathbb{Q}_p,\ x_p \in \mathbb{Z}_p \text{ for a.e. } p\}.$$

Indeed any $x \in \mathbb{Q}$ belong to $\mathbb{Z}_p$ for all but finitely many p, namely the p which do not divide the denominator of p.

Definition 3.1. *The ring $\mathbb{A}$ is called the ring of adèles of $\mathbb{Q}$. It factors as*

$$\mathbb{A} := \mathbb{R} \times \mathbb{A}_f$$

where

$$\mathbb{A}_f := \prod_p{}' \mathbb{Q}_p = \{(x_p)_p,\ x_p \in \mathbb{Q}_p,\ x_p \in \mathbb{Z}_p \text{ for a.e. } p\} \subset \mathbb{Q}_{\mathcal{V}_f}$$

is the restricted product of the $\mathbb{Q}_p$ with respect to the sequence of subsets $(\mathbb{Z}_p)_p$ and is called the ring of finites adèles of $\mathbb{Q}$.

More generally, for $S \subset \mathcal{V}$ a set of places we denote by $\mathbb{A}_S$ and $\mathbb{A}^{(S)} = \mathbb{A}_{\mathcal{V}-S}$

$$\mathbb{A}_S = \prod_{v \in S}{}' \mathbb{Q}_v = \{(x_v)_{v \in S},\ x_v \in \mathbb{Q}_v,\ x_p \in \mathbb{Z}_p \text{ for a.e. } p \in S\} \subset \mathbb{Q}_S,$$

and

$$\mathbb{A}^{(S)} = \prod_{v \notin S}{}' \mathbb{Q}_v \{(x_v)_{v \in S},\ x_v \in \mathbb{Q}_v,\ x_p \in \mathbb{Z}_p \text{ for a.e. } p \notin S\} \subset \mathbb{Q}_{\mathcal{V}-S}.$$

For instance, $\mathbb{A}_{\mathcal{V}} = \mathbb{A}$, $\mathbb{A}_p = \mathbb{Q}_p$, $\mathbb{A}_{\mathcal{V}_f} = \mathbb{A}^{(\infty)} = \mathbb{A}_f$. These are subrings of the corresponding products for the pointwise addition and multiplication. Moreover $\mathbb{A}^S$ and $\mathbb{A}^{(S)}$ embeds into as $\mathbb{A}$ as $\mathbb{A}$-modules via

$$\mathbb{A}_S \simeq \{(x_v)_v \in \mathbb{A},\ x_v = 0 \ \text{ for all } v \notin S\}, \tag{2.1}$$

$$\mathbb{A}^{(S)} = \mathbb{A}_{\mathcal{V}-S} = \{(x_v)_v \in \mathbb{A},\ x_v = 0 \ \text{ for all } v \in S\}.$$

Remark 2.1. The set $\mathbb{A}$ is indeed a subring of $\mathbb{Q}_{\mathcal{V}}$ for the pointwise addition and multiplication: if $(x_v)_v$ and $(y_v)_v$ are such that $x_p, y_p \in \mathbb{Z}_p$ a.e. p, then $x_p + y_p,\ x_p y_p \in \mathbb{Z}_p$ a.e. p.

Remark 2.2. The field $\mathbb{Q}$ embeds into $\mathbb{A}_S$ and $\mathbb{A}^{(S)}$ via the diagonal embeddings δ_S and $\delta^{(S)} = \delta_{\mathcal{V}-S}$ giving these rings the structure of $\mathbb{Q}$-algebras.

2.1. The adelic topology. We now equip these rings with an adequate restricted product topology

DEFINITION 3.2. *The adelic topology on $\mathbb{A}_S \subset \mathbb{Q}_S$ is the restriction to $\mathbb{A}_S$ of the product topology on $\mathbb{Q}_S$. A basis of open neighborhoods is composed of the subsets of the shape*

$$\Omega = \prod_{v \in S} \Omega_v \subset \mathbb{A}_S,\ \Omega_v \subset \mathbb{Q}_v \text{ open },\ \Omega_p = \mathbb{Z}_p \text{ a.e. } p \in S\ .$$

EXERCISE 3.1. Prove that addition and multiplication are continuous for the adelic topology. Prove that the adelic topology on $\mathbb{A}_S$ is the one obtained by restricting of the adelic topology on $\mathbb{A}$ to the image of the embedding (2.1) and that this image is closed in $\mathbb{A}$.

With this definition the strong approximation theorem is equivalent to

THEOREM. *For any place $v_0 \in \mathcal{V}$, $\mathbb{Q}$ (embeded by $\delta^{(v_0)}$) is dense in $\mathbb{A}^{(v_0)}$. In particular $\mathbb{Q}$ is dense in $\mathbb{A}_f$.*

This result is optimal since we have

THEOREM. *$\mathbb{Q}$ is discrete in $\mathbb{A}$.*

PROOF. Since additive translations are homeomorphisms it is sufficient to show that 0 is isolated: there exists an open set $\Omega = \prod_v \Omega_v$ such that

$$\mathbb{Q} \cap \Omega = \{0\}.$$

One take

$$\Omega = [-1/2, 1/2] \times_p \mathbb{Z}_p$$

so that if $x \in \mathbb{Q} \cap \Omega$ we have

$$|x|_\infty \leqslant 1/2,\ x \in \mathbb{Z}_p \text{ for all } p \Leftrightarrow |x|_\infty \leqslant 1/2,\ x \in \mathbb{Z}$$

and therefore $x = 0$. □

2.2. The ring of adelic integers. Let $\widehat{\mathbb{Z}}$ be the product of all p-adic integers

$$\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \subset \mathbb{A}_f$$

this is an open subring of $\mathbb{A}_f$.

PROPOSITION 3.1. *The ring $\widehat{\mathbb{Z}}$ is open, compact and locally compact.*

PROOF. $\widehat{\mathbb{Z}}$ is clearly open (since the $\mathbb{Z}_p$ are open). It is also closed being the complement of the union of the open sets (because $\mathbb{Z}_p$ is closed in $\mathbb{Q}_p$)

$$(\mathbb{Q}_p - \mathbb{Z}_p) \prod_{p' \neq p}{}' \mathbb{Q}_{p'}.$$

Observe that the topology induced by the inclusion $\widehat{\mathbb{Z}} \subset \mathbb{A}_f$ is precisely the *product (or Tychonoff)* topology on $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$: the topology whose basis of open subsets is given by the sets of the shape

$$\Omega = \prod_p \Omega_p,\ \Omega_p \subset \mathbb{Z}_p \text{ open },\ \Omega_p = \mathbb{Z}_p \text{ a.e. } p\ .$$

The rings $\mathbb{Z}_p$ being compact, and locally compact, by Tychonoff theorem $\widehat{\mathbb{Z}}$ is compact, locally compact. Let $R \subset \mathbb{A}_f$ be a compact subring and let R_p be its projection to $\mathbb{Q}_p$; this is a compact subring of $\mathbb{Q}_p$ hence is contained in $\mathbb{Z}_p$. $\square$

Since translations are homeomorphism we obtain that

THEOREM 3.3. *$\mathbb{A}_f$ and $\mathbb{A}$ are locally compact topological rings.*

REMARK 2.3. Observe that if S is infinite, the product ring $\mathbb{Q}_S$ is NOT locally compact. This is another reason why the adeles equipped with adelic topology is the right space to consider.

Since $\widehat{\mathbb{Z}}$ is open, by the strong approximation theorem for $\mathbb{A}_f$ one has

PROPOSITION 3.2. *One has the decompositions*

$$\mathbb{A}_f = \mathbb{Q} + \widehat{\mathbb{Z}},\ \mathbb{A} = \mathbb{Q} + \mathbb{R} + \widehat{\mathbb{Z}}.$$

REMARK 2.4. In the above expression, $\mathbb{Q}$ should be understood as $\delta_{\mathcal{V}_f}(\mathbb{Q})$ for the first equality and $\delta_{\mathcal{V}}(\mathbb{Q})$ for the second. In the second $\mathbb{R} = \mathbb{A}_\infty = (\mathbb{R}, 0, 0\cdots)_v$ and $\widehat{\mathbb{Z}} = (0, \mathbb{Z}_2, \mathbb{Z}_3, \cdots)$.

PROOF. Since $\mathbb{Q}$ is dense and $\widehat{\mathbb{Z}}$ is open in $\mathbb{A}_f$, the union of translates of $\widehat{\mathbb{Z}}$ by the elements of $\mathbb{Q}$ covers all of $\mathbb{A}_f$: $\mathbb{Q} + \widehat{\mathbb{Z}} = \mathbb{A}_f$; the second result follows immediately. $\square$

Since $\mathbb{Q} \cap \widehat{\mathbb{Z}} = \mathbb{Z}$ we obtain the following version of the

THEOREM (Chinese Reminder Theorem). *$\mathbb{Z}$ is dense in $\widehat{\mathbb{Z}}$ for the adelic topology.*

EXERCISE 3.2. Prove this result directly.

EXERCISE 3.3. Prove that $\widehat{\mathbb{Z}}$ is maximal for the compactness property: more precisely any compact subring of $\mathbb{A}_f$ is contained in $\widehat{\mathbb{Z}}$.

EXERCISE 3.4. Prove that as a group $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$.

EXERCISE 3.5. Prove that if $K_f \subset \mathbb{A}_f$ is a compact subset, there exist $m \in \mathbb{Z} - \{0\}$ such that K_f is a finite disjoint unions of points and of translates of $m\widehat{\mathbb{Z}}$.

EXERCISE 3.6. Prove the finer decomposition

$$\mathbb{A} = \mathbb{Q} + [-1/2, 1/2] + \widehat{\mathbb{Z}} \tag{2.2}$$

Since $\mathbb{Q}$ is discrete in $\mathbb{A}$ it is also closed and we may consider the quotient $\mathbb{Q}\backslash\mathbb{A}$ space; equipped with the quotient topology this quotient is locally compact separated space (Cf. the Appendix). Since $\widehat{\mathbb{Z}}$ is compact the same is true for the quotient $(\mathbb{Q} + \widehat{\mathbb{Z}})\backslash\mathbb{A}$. We have the following

THEOREM 3.4. *The quotient $\mathbb{Q}\backslash\mathbb{A}$ is compact. We have an homeomorphism*

$$(\mathbb{Q} + \widehat{\mathbb{Z}})\backslash\mathbb{A} \simeq \mathbb{R}/\mathbb{Z} \simeq S^1$$

PROOF. The images of

$$\widehat{\mathbb{Z}} \to \mathbb{Q}\backslash\mathbb{A},\ [-1/2, 1/2] \to \mathbb{Q}\backslash\mathbb{A}$$

under the (continuous) projection map are compact images and their sum is $\mathbb{Q}\backslash\mathbb{A}$ (by (2.2)). We leave the second statement as an exercise (observe that $\mathbb{Q} \cap \widehat{\mathbb{Z}} = \mathbb{Z}$). $\square$

3. The group of Idèles

The group of idèles, $\mathbb{A}^\times$ is the multiplicative group of invertible elements (or units) of the ring $\mathbb{A}$. It decompose as the *restricted product* of the multiplicative groups $(\mathbb{Q}_v^\times)_{v\in\mathcal{V}}$ with respect to the sequence of subgroups $(\mathbb{Z}_p^\times)_p$:

$$\mathbb{A}^\times = \{(x_v)_v \in \mathbb{A},\ x_v \in \mathbb{Q}_v^\times,\ x_p \in \mathbb{Z}_p^\times \text{ for a.e. } p\} = \prod_v{}' \mathbb{Q}_v^\times \subset \mathbb{Q}_\mathcal{V}^\times = \prod_{v\in\mathcal{V}} \mathbb{Q}_v^\times$$

In the same way, for $S \subset \mathcal{V}$, the group of units of $\mathbb{A}_S$ is given by

$$\mathbb{A}_S^\times = \{(x_v)_{v\in S},\ x_v \in \mathbb{Q}_v^\times,\ x_p \in \mathbb{Z}_p^\times \text{ for a.e. } p \in S\} = \prod_{v\in S}{}' \mathbb{Q}_v^\times \subset \mathbb{Q}_S^\times = \prod_{v\in S} \mathbb{Q}_v^\times.$$

In particular $\mathbb{A}_f^\times = \mathbb{A}_{\mathcal{V}_f}^\times = \mathbb{A}^{\times(\infty)}$ is called the group of finite ideles. We will realize the group $\mathbb{A}_S^\times$ as a subgroup of $\mathbb{A}^\times$ via

$$\mathbb{A}_S^\times \simeq \{(x_v)_v \in \mathbb{A}^\times,\ x_v = 1\ \forall\ v \notin S\};$$

3.1. The topology of idèles. What is called the *adelic topology* on the group of idèles $\mathbb{A}^\times$ or any $\mathbb{A}_S^\times$ is NOT a priori the restriction of the adelic topology relative the inclusion $\mathbb{A}^\times \subset \mathbb{A}$ (or $\mathbb{A}_S^\times \subset \mathbb{A}_S$).

Definition 3.3. *The adelic topology on $\mathbb{A}^\times$ (and similarly for $\mathbb{A}_S^\times$) is either (prove that the two definitions are the same)*

- *The restriction of the product topology relative to the inclusion*

$$\prod_v{}' \mathbb{Q}_v^\times \subset \prod_v \mathbb{Q}_v^\times.$$

  In other terms a basis of open sets for $\mathbb{A}^\times$ is given by set of the shape

$$\Omega = \prod_v \Omega_v,\ \Omega_v \subset \mathbb{Q}_v^\times \text{ open},\ \Omega_p = \mathbb{Z}_p^\times \text{ a.e. } p$$

  (observe that $\mathbb{Z}_p^\times$ is an open compact subgroup of $\mathbb{Q}_p^\times$ equipped with the p-adic topology).
- *The restriction of the adelic topology on $\mathbb{A}^2$ when $\mathbb{A}^\times$ is realized as the closed subset*

$$\mathbb{A}^\times \simeq \{(x,y) \in \mathbb{A}^2,\ xy = 1\} \subset \mathbb{A}^2$$

  via the map $x \in \mathbb{A}^\times \mapsto (x, x^{-1}) \in \mathbb{A}^2$.

Theorem 3.5. *With this topology, $\mathbb{A}^\times$ is a locally compact topological group (multiplication and inversion are continuous) of which the $\mathbb{A}_S^\times$ are closed subgroups and $\widehat{\mathbb{Z}}^\times$ is an open compact subgroup of $\mathbb{A}_f^\times$.*

Proof. Since component-wise multiplication on $\mathbb{A}^2$ and the involution $(x,y) \to (y,x)$ are continuous multiplication and inversion are continuous on $\mathbb{A}^\times$ (taking the second definition of the adelic topology). Since $\mathbb{Z}_p^\times$ is an open and compact subgroup of $\mathbb{Q}_p^\times$, $\widehat{\mathbb{Z}}^\times$ equipped with the adelic topology (which is nothing else than the Tychonoff topology) is open, compact and locally compact of subgroup $\mathbb{A}_f$; from this the local compactness of $\mathbb{A}_f^\times$ and $\mathbb{A}^\times$ follow by a translation argument. □

EXERCISE 3.7. Prove that the relative topology on $\mathbb{A}^\times \subset \mathbb{A}$ is not the adelic topology defined above

EXERCISE 3.8. Prove that $\mathbb{Z}_p^\times$ and $\widehat{\mathbb{Z}}$ are maximal compact subgroups of $\mathbb{Q}_p^\times$ and $\mathbb{A}_f^\times$ respectively: any compact subgroup is contained in it.

EXERCISE 3.9. Given $q_f \in \mathbb{A}_f^\times \cap \widehat{\mathbb{Z}}$, let

$$K(q_f) := \{x_f \in \widehat{\mathbb{Z}}^\times,\ x_f \,(\mathrm{mod}\, q_f) = 1 \in \widehat{\mathbb{Z}}/q_f\widehat{\mathbb{Z}}\}.$$

Prove that $K(q_f)$ is an open compact subgroup of $\widehat{\mathbb{Z}}^\times$ which depends only on the positive integer

$$q := \prod_p p^{v_p(q_f)}.$$

This group also noted $K(q)$ is called the *principal congruence subgroup* of level q; prove that as q varies, these groups form a basis of open neighborhoods of 1 in $\mathbb{A}_f^\times$. Prove that

$$\widehat{\mathbb{Z}}^\times/K(q_f) \simeq (\mathbb{Z}/q\mathbb{Z})^\times.$$

Let us now discuss how much "space" the subgroup of rational elements $\mathbb{Q}^\times$ occupies into $\mathbb{A}^\times$.

PROPOSITION 3.3. *The group $\mathbb{Q}^\times$ is discrete in $\mathbb{A}^\times$. One has the following decompositions*

$$\mathbb{A}_f^\times = \mathbb{Q}^\times\widehat{\mathbb{Z}}^\times,\ \mathbb{A}^\times = \mathbb{Q}^\times\mathbb{R}^\times\widehat{\mathbb{Z}}^\times$$

and

$$\mathbb{Q}^\times\backslash\mathbb{A}_f/K(q_f) \simeq (\mathbb{Z}/q\mathbb{Z})^\times.$$

On the other hand $\mathbb{Q}^\times$ is not copmpact in $\mathbb{A}^\times$. To see the obstruction we introduce

DEFINITION 3.4. *The* adelic modulus *(or adelic absolute value) is the map given by the converging product*

$$|\cdot|_{\mathbb{A}} : x = (x_v)_v \in \mathbb{A}^\times \mapsto |x|_{\mathbb{A}} := \prod_v |x_v|_v \in \mathbb{R}_{>0}.$$

Indeed this is well defined since for since for a.e. p, $|x_p|_p \leqslant 1$. Notice that $|\cdot|_{\mathbb{A}}$ is identically 0 on $\mathbb{A} - \mathbb{A}^\times$ while on $\mathbb{A}^\times$ the above infinite product is converging (to a non-zero limit) since for a.e. p, $|x_p|_p \leqslant 1$.

EXERCISE 3.10. Prove that $|\cdot|_{\mathbb{A}} : \mathbb{A}^\times \to \mathbb{R}_{>0}$ is a continuous group homomorphism.

We denote by $\mathbb{A}^1$ the group of ideles of modulus 1 (the kernel of $|\cdot|_{\mathbb{A}}$): this is a closed subgroup. We have

PROPOSITION 3.4 (Product formula). *We have*

$$\mathbb{A}^1 = \mathbb{Q}^\times \times \widehat{\mathbb{Z}}.$$

In particular for any $x_{\mathbb{Q}} \in \mathbb{Q}^\times$ one has the

$$(Product\ formula)\ |x|_{\mathbb{A}} = |x|_\infty \prod_p |x|_p = 1$$

Proof. If is obvious that $\widehat{\mathbb{Z}}^\times \subset \mathbb{A}^1$ while for $x_\mathbb{Q} \in \mathbb{Q}^\times$

$$|x|_\mathbb{A} = |x|_\infty \prod_p |x|_p = \prod_p p^{v_p(x)} \prod_p p^{-v_p(x)} = 1.$$

Given $x = (x_\mathbb{R}, x_f) \in \mathbb{R}^\times \times \mathbb{A}_f^\times$ of modulus 1, we have $x_\mathbb{R} = \pm|x_f|_\mathbb{A}^{-1} \in \mathbb{Q}^\times$, therefore up to multiplying x by a rational number we may assume that $x = (1, x_f)$ and $|x_f|_\mathbb{A} = 1$ which is equivalent to $|x_p|_p = 1$ for every p or in other terms $(x_p)_p \in \widehat{\mathbb{Z}}^\times$. □Observe that

$$|\mathbb{Q}^\times\backslash\mathbb{A}^\times|_\mathbb{A} = |\mathbb{R}^\times|_\mathbb{A} = \mathbb{R}_{>0}$$

is not compact, therefore

The quotient $\mathbb{Q}^\times\backslash\mathbb{A}^\times$ is not compact.

This is the only obstruction:

Theorem 3.6. *The group $\mathbb{Q}^\times$ is discrete in $\mathbb{A}^1$ and the quotient $\mathbb{Q}^\times\backslash\mathbb{A}^1$ is compact.*

Proof. Since $\mathbb{Q}^2 \subset \mathbb{A}^2$ is discrete it follows (from the definition of the adelic topology and the fact that $\mathbb{A}^1$ is closed that $\mathbb{Q}^\times$ is discrete in $\mathbb{A}^1$. The compactness of $\mathbb{Q}^\times\backslash\mathbb{A}^1$ follows from the decomposition $\mathbb{A}^1 = \mathbb{Q}^\times \times \widehat{\mathbb{Z}}$ and the compactness of $\widehat{\mathbb{Z}}^\times$. □

Finally we observe that strong approximation does not hold for the ideles:

Exercise 3.11. Prove that

$$\mathbb{Q}^\times\backslash\mathbb{A}_f^\times/K(q_f) \simeq (\mathbb{Z}/q\mathbb{Z})^\times$$

where $q = \prod_p p^{v_p(q_f)}$. In particular $\mathbb{Q}^\times$ is not dense in $\mathbb{A}_f^\times$.

Remark 3.1. As we will see in greater generality later the decomposition $\mathbb{A}_f^\times = \mathbb{Q}^\times \times \widehat{\mathbb{Z}}^\times$ is equivalent to the fundamental

Theorem 3.7. *The ring $\mathbb{Z}$ is a principal ideal ring: any ideal $0 \neq I \subset \mathbb{Z}$ is of the shape $m\mathbb{Z}$ for some $m \in \mathbb{Z}$.*