CHAPTER 4

Higher dimensional adelic structures

In this chapter we discuss and describe the structure of higher dimensional adelic objects; the most basic one being the free $\mathbb{A}$-module $\mathbb{A}^n$ equipped with the adelic product topology and subsets of it. We first recall the following:

1. Finitely generated modules over principal ideal rings

Let R be a (commutative) ring; we recall that an R-module L is finitely generated if there exists some finite set $\{\mathbf{e}_i\} \subset L$ such that

$$L = \sum_i R\mathbf{e}_i.$$

An R-module is free is there exists a finite generating set $\{\mathbf{e}_i\} \subset L$ such that any element of L can be written in a unique way as a (finite) R-linear combination of elements of the $\mathbf{e}_i$: the set $\{\mathbf{e}_i\} \subset L$ is then called a (free) R-basis of L and this is written

$$L = \bigoplus_i R\mathbf{e}_i.$$

A basis is not unique but its cardinality depends only on L and is called the R-rank of L, $\mathrm{rk}_R(L)$. When R is a principal ideal ring, finitely generated R-modules are well understood:

Theorem 4.1. *Let L be a finitely generated R-module; L decompose as a direct sum of a free R-module and of a torsion R-module*

$$L = L' \oplus L_{tors}$$

where

$$L_{tors} = \{x \in L,\ exists r \in R,\ r.x = 0_L\}$$

denote the torsion part of L and $L' \subset L$ is free; in particular L is free iff $L_{tors} = 0$. The free module L' is not unique but is rank depends only on L is is again called the R-rank of L, $\mathrm{rk}_R(L)$.

If $L' \subset L$ is a submodule, L' is finitely generated and one has

$$L'_{tors} \subset L_{tors},\ \mathrm{rk}_R(L') \leqslant \mathrm{rk}_R(L).$$

In particular if L is free then L' is free and there exist a basis of L

$$\{\mathbf{e}_1, \cdots, \mathbf{e}_{r'}, \cdots \mathbf{e}_r\}$$

and elements of R, $d_{r'}|d_{r'-1}|\cdots|d_1 \neq 0$ such that

$$\{d_1\mathbf{e}_1, \cdots, d_{r'}\mathbf{e}_{r'}\}$$

is a basis of L'.

2. Free $\mathbb{A}$-module

For any $n \geqslant 1$, we consider the free $\mathbb{A}$-module

$$\mathbb{A}^n = \{\mathbf{x} = (x_1, \cdots, x_n),\ x_i \in \mathbb{A}\}.$$

One can check that this can be written in restricted product form

$$\mathbb{A}^n = \{(\mathbf{x}_v)_{v \in \mathcal{V}},\ \mathbf{x}_v \in \mathbb{Q}_v^n,\ \mathbf{x}_p \in \mathbb{Z}_p^n \text{ for a.e. } p\} = \prod_v{}' \mathbb{Q}_v^n$$

the restricted product being wrt the sequence of free modules $(\mathbb{Z}_p^n)_p$. For any $V \subset \mathcal{V}$, we can form similarly the $\mathbb{A}_V$-modules

$$\mathbb{A}_V^n = \prod_{v \in V}{}' \mathbb{Q}_v^n \simeq \{(\mathbf{x}_v)_v \in \mathbb{A}^n,\ \mathbf{x}_v = 0 \text{ for all } v \notin V\}$$

in which $\mathbb{Q}^n$ embeds as a $\mathbb{Q}$-vecor space via the map δ_V. We equip $\mathbb{A}^n$ with the adelic product topology and

Proposition 4.1. *The following properties hold:*

- $\mathbb{A}^n$ *is a topological* $\mathbb{A}$*-module (addition and multiplication by scalars are continuous).*
- *For any* $V \subset \mathcal{V}$ *,* $\mathbb{A}_V^n \subset \mathbb{A}^n$ *is closed.*
- $\delta_{\mathcal{V}} : \mathbb{Q}^n \to \mathbb{A}^n$ *is discrete and the quotient* $\mathbb{Q}^n \backslash \mathbb{A}^n = (\mathbb{Q} \backslash \mathbb{A})^n$ *is compact.*
- *(Weak approximation) For any finite* $V \subset \mathcal{V}$*,* $\delta_V : \mathbb{Q}^n \to \mathbb{A}_V^n$ *is dense.*
- *(Strong approximation) For any* $v \in \mathcal{V}$*,* $\delta^{(v)} : \mathbb{Q}^n \to (\mathbb{A}^{(v)})^n$ *is dense. In particular* $\delta_f : \mathbb{Q}^n \to \mathbb{A}_f^n$ *is dense.*

Proposition 4.2. *Let* $\mathcal{B} = \{\mathbf{e}_i,\ i = 1 \cdots n\}$ *be a* $\mathbb{Q}$*-basis of* $\mathbb{Q}^n$*, then* $\mathcal{B}$ *is a free basis of the* $\mathbb{A}$*-module* $\mathbb{A}^n$ *(and of any* $\mathbb{A}_V^n$ *for* $V \subset \mathcal{V}$*):*

$$\mathbb{A}^n = \bigoplus_i \mathbb{A}\mathbf{e}_i.$$

Moreover we have

$$\mathbb{A}^n = \{(\mathbf{x}_v)_v \in \prod_v \mathbb{Q}_v,\ \mathbf{x}_p \in L_p \text{ a.e. } p\}$$

where L_p *is the free* $\mathbb{Z}_p$*-module (and the closure of* $\sum_i \mathbb{Z}\mathbf{e}_i$ *in* $\mathbb{Q}_p^n$*)*

$$L_p = \bigoplus_i \mathbb{Z}_p \mathbf{e}_i \subset \mathbb{Q}_p^n.$$

In other terms $\mathbb{A}^n$ *is the restricted product of the* $\mathbb{Q}_v^n$ *wrt to the sequence* $(L_p)_p$*. Moreover the sequence* $(L_p)_p$ *has the property that*

$$L_p = \mathbb{Z}_p^n$$

for a.e. p*.*

Proof. □

Remark 2.1. Observe that these two representations as restricted product are indeed the same because (chasing denominators), there always exists $m \in \mathbb{Z} - \{0\}$ such that

$$m \sum_i \mathbb{Z}\mathbf{e}_i \subset \mathbb{Z}^n \subset \frac{1}{m} \sum_i \mathbb{Z}\mathbf{e}_i$$

which implies that for p not dividing m

$$L_p = \mathbb{Z}_p^n.$$

3. Local-global principles for lattices

DEFINITION 4.1. *A lattice $L \subset \mathbb{Q}^n$ is a finitely generated $\mathbb{Z}$-module generating $\mathbb{Q}^n$ as a $\mathbb{Q}$-vector space. we denote by $\mathcal{L}(\mathbb{Q}^n)$ the set of all lattices in $\mathbb{Q}^n$*

This definition admits the following "local" counterpart:

DEFINITION 4.2. *For $v \in \mathcal{V}$, let $\mathbb{Z}_v$ be the closure*[1] *of $\mathbb{Z}$ in $\mathbb{Q}_v$; a lattice $L_v \subset \mathbb{Q}_v^n$ is a finitely generated $\mathbb{Z}_v$-module generating $\mathbb{Q}_v^n$ as a $\mathbb{Q}_v$-vector space. We denote by $\mathcal{L}(\mathbb{Q}_v^n)$ the set of all lattices in $\mathbb{Q}_v^n$.*

REMARK 3.1. Since $\mathbb{Z}$ (resp. $\mathbb{Z}_v$) is a principal ideal ring, a lattice L in $\mathbb{Q}^n$ (resp. $\mathbb{Q}_v^n$) is precisely a free $\mathbb{Z}$ (resp. $\mathbb{Z}_v$)-module of rank n

$$L = \bigoplus_{i=1\cdots n} \mathbb{Z}\mathbf{e}_i$$

where $\mathcal{B} = \{\mathbf{e}_i\}$ is a basis of $\mathbb{Q}^n$.

Let $L \subset \mathcal{L}(\mathbb{Q}^n)$ be a rational lattice and let $\mathcal{B} = \{\mathbf{e}_i\}$ be a basis of $\mathbb{Q}^n$ generating L. In the previous section we have associated to such a lattice the sequence of $\mathbb{Q}_v$-lattices $(L_v)_v$,

$$L_v = \sum_i \mathbb{Z}_v \mathbf{e}_i \subset \mathbb{Q}_v^n;$$

alternatively L_v is defined as the closure of L in $\mathbb{Q}_v^n$. This sequence satisfies the following property: *one has*

$$L_p = \mathbb{Z}_p^n \textit{ for a.e. } p.$$

Observe also that the product $\prod_p L_p$ viewed as a subset of $\mathbb{A}_f^n$ equals

$$\prod_p L_p = \sum_i \widehat{\mathbb{Z}}\mathbf{e}_i = \widehat{L}$$

the closure of L in $\mathbb{A}_f^n$.

This collection of "local" data $(L_v)_v$ suffice to recover the lattice L: in fact it is sufficient to consider only the place at ∞ since $L_\infty = L \subset \mathbb{R}^n$. On the other hand knowing L_p for a given p is not sufficient to distinguish L from other rational lattices but the data, $(L_p)_p$, for all the finite places does:

THEOREM 4.2 (Local-global principle for lattices). *Let $\mathcal{L}(\mathbb{A}_f^n)$ denote restricted product*

$$\prod_p{}' \mathcal{L}_n(\mathbb{Q}_p) = \{(L_p)_p,\ L_p \in \mathcal{L}_n(\mathbb{Q}_p),\ \textit{such that } L_p = \mathbb{Z}_p^n \ \textit{ for a.e. } p\};$$

the map

$$\begin{array}{ccc} \mathcal{L}(\mathbb{Q}^n) & \mapsto & \prod'_{p\in\mathcal{P}} \mathcal{L}(\mathbb{Q}_p^n) \\ L & \mapsto & (L_p)_p \end{array}$$

[1] for $v = \infty$, $\mathbb{Q}_v = \mathbb{R}$ and $\mathbb{Z}_v = \mathbb{Z}$

is $1-1$ *and its converse is the map*

$$(L_p)_p \mapsto L = \bigcap_p \mathbb{Q}^n \cap L_p.$$

PROOF. We have seen already that $L \mapsto (L_p)_p$ maps $\mathcal{L}(\mathbb{Q}^n)$ to $\prod_p{}' \mathcal{L}(\mathbb{Q}_p^n)$. Let us show that it has a converse is given by the map

$$(L_p)_p \mapsto \bigcap_p \mathbb{Q}^n \cap L_p.$$

Given any $(L_p)_p \in \prod_p{}' \mathcal{L}_n(\mathbb{Q}_p)$, let $L' = \bigcap_p \mathbb{Q}^n \cap L_p \subset \mathbb{Q}^n$, we first show that L' is a lattice. Let S be the finite set of primes such that $L_p \neq \mathbb{Z}_p^n$. For any such p there exist $\alpha_p = \alpha_p(L_p) \geqslant 0$ such that

$$p^{\alpha_p}\mathbb{Z}_p^n \subset L_p \subset p^{-\alpha_p}\mathbb{Z}_p^n$$

and for $N = \prod_{p\in S} p^{\alpha_p}$ we have

$$N\mathbb{Z}^n = \bigcap_p \mathbb{Q}^n \cap N\mathbb{Z}_p^n \subset L' \subset \bigcap_p \mathbb{Q}^n \cap N^{-1}\mathbb{Z}_p^n = N^{-1}\mathbb{Z}^n$$

so L' is a lattice.

Suppose now that the $(L_p)_p$ are obtained as the closure of some lattice $L \subset \mathbb{Q}^n$: we will show that $L' = L$. Observe that L_p is the set of elements in $\mathbb{Q}_p^n$ whose coordinates in the basis $\{\mathbf{e}_i\}$ are p-adic integers. Therefore L' is the set of element in $\mathbb{Q}^n$ whose coordinates in the same basis are p-adic integers for every p: meaning integers. □

4. Adelic points of a vector space

Let

$$K = \bigoplus_{i=1,\cdots,n} \mathbb{Q}\mathbf{e}_i$$

be a general n-dimensional $\mathbb{Q}$-vector space with a choice of a $\mathbb{Q}$-basis $\mathcal{B} = \{\mathbf{e}_i,\ i = 1\cdots n\}$; we denote by

$$K_{\mathcal{V}} = K(\mathbb{A}) = \bigoplus \mathbb{A}\mathbf{e}_i \simeq \mathbb{A}^n$$

the free rank-n, $\mathbb{A}$-module with basis the set $\mathcal{B}$. The vector space K embeds into $K(\mathbb{A})$ in the obvious way. In the same way we define for any $S \subset \mathcal{V}$,

$$K(\mathbb{A}_S) = \bigoplus \mathbb{A}_S\mathbf{e}_i \simeq \mathbb{A}_S^n$$

in which K embeds via δ_S. In particular for $v \in \mathcal{V}$

$$K_v = K(\mathbb{Q}_v) = \bigoplus \mathbb{Q}_v\mathbf{e}_i \simeq \mathbb{Q}_v^n.$$

For $v = p$, we denote by

$$L_{\mathcal{B},p} = \sum_i \mathbb{Z}_p\mathbf{e}_i \subset K_p$$

the free $\mathbb{Z}_p$-module generated by $\mathcal{B}$. With these notations we have

$$K(\mathbb{A}) = \prod_v{}' K_v = \{(\mathbf{x}_v)_{v\in\mathcal{V}},\ \mathbf{x}_v \in K_v,\ \mathbf{x}_p \in L_{\mathcal{B},p} \text{ for a.e. } p\}$$

and

$$K(\mathbb{A}_S) \simeq \{(\mathbf{x}_v)_{v\in\mathcal{V}} \in K(\mathbb{A}),\ \mathbf{x}_v = 0 \text{ for } v \notin V\}$$

4.1. Topology. The space $K(\mathbb{A})$ is obviously isomorphic to $\mathbb{A}^n$ as a $\mathbb{A}$-module (by chosing some basis $\mathcal{B}$) is therefore inherit the adelic topology. This topology does not depend on the choice of $\mathcal{B}$, because two basis are transform into each other by a $\mathbb{Q}$-linear map which is an homeomorphism. The following are immediate consequences of the case $K = \mathbb{Q}$:

THEOREM 4.3. *For $n \geqslant 1$, one has*

- *For any $V \subset \mathcal{V}$, the $K(\mathbb{A}_V)$ embeds as a closed subset of $K(\mathbb{A})$ (the subset of $(\mathbf{x}_v)_v \in K(\mathbb{A})$ such that $\mathbf{x}_v = 0$ for all $v \notin V$).*
- *(Discretness) $K \hookrightarrow K(\mathbb{A})$ (embedded diagonally via $\delta_{\mathcal{V}}$) is discrete.*
- *(Weak approximation) For any $V \subset \mathcal{V}$ finite, $\delta_V(K)$ is dense in $K(\mathbb{A}_V)$.*
- *(Strong approximation) For any $v \in \mathcal{V}$, $\delta^{(v)}(K)$ is dense in $K(\mathbb{A}^{(v)})$.*

4.2. Tensor products. These constructions can be made more systematic with using the notion of tensor product of two $\mathbb{Q}$ vector spaces (cf. Appendix): given K, L two $\mathbb{Q}$-vector spaces (not necessarily finite dimensional), there exists a (unique up to unique isomorphism) $\mathbb{Q}$-vector space $K \otimes_{\mathbb{Q}} L$ and a bilinear map

$$\otimes : (u,v) \in K \times L \to u \otimes v \in K \otimes_{\mathbb{Q}} L$$

such that for any bilinear map into another $\mathbb{Q}$-vector space

$$f : (u,v) \in K \times L \to f(u,v) \in M$$

there exists a linear map $\tilde{f} : K \otimes L \to M$ such that

$$\tilde{f}(u \otimes v) = f(u,v).$$

It follows from these properties that the map $\otimes$ is injective and moreover, if $\mathcal{B} = \{u_i\}_{i\in I}$, $\mathcal{B}' = \{v_j\}_{j\in J}$ are bases of K and L,

$$\mathcal{B} \otimes \mathcal{B}' = \{u_i \otimes v_j\}_{(i,j)\in I\times J}$$

is a basis of $K \otimes L$.

EXERCISE 4.1. Check that when one applies this construction to K and $L = \mathbb{Q}_v, \mathbb{A}_V$ or $\mathbb{A}$ one obtains respectively

$$K \otimes \mathbb{Q}_v = K(\mathbb{Q}_v),\ K \otimes \mathbb{A}_V = K(\mathbb{A}_V),\ K \otimes \mathbb{A} = K(\mathbb{A})$$

We also have the following:

PROPOSITION 4.3. *Let $f : V \to W$ be a linear map of $\mathbb{Q}$-vector spaces, then f extends to an $\mathbb{A}$-linear map on $f : V(\mathbb{A}) \to W(\mathbb{A})$ which is continuous for the adelic topology.*

4.3. Lattices in vector spaces. The discussion of Section 3 generalize to the setting of finite dimensional vector spaces in an obvious way:

DEFINITION 4.3. *Let V be either a $\mathbb{Q}$ or a $\mathbb{Q}_v$ finite dimensional vector space. A lattice $L \subset V$ is a finitely generated $\mathbb{Z}$ or $\mathbb{Z}_v$-module generating V as a vector space. we denote by $\mathcal{L}(V)$ the set of all lattices in V*

We have the following extension of the local-global principle for lattices:

THEOREM 4.4. *Let V be a n-dimensional $\mathbb{Q}$-vector space and $L_0 \subset V$ a fixed lattice. For any lattice*

$$L = \sum_{i=1}^{n} \mathbb{Z}\mathbf{e}_i \in \mathcal{L}(V)$$

we denote by

$$L_v = \overline{L} \sum_{i=1}^{n} \mathbb{Z}_v \mathbf{e}_i \in \mathcal{L}(V_v)$$

the corresponding local lattice. The map

$$L \mapsto (L_p)_{p \in \mathcal{V}_f}$$

is a bijection between $\mathcal{L}(V)$ and

$$\prod_p{}' \mathcal{L}(V_p) = \{(L_p)_p,\ L_p \in \mathcal{L}(V_p),\ L_p = L_{0,p} \text{ for a.e. } p\}$$

whose inverse is given by

$$(L_p)_p \mapsto L = \cap_p V \cap L_p \subset V.$$

5. Adelic points of an algebraic variety

Let

$$P_1, \cdots, P_r \subset \mathbb{Q}[X_1, \cdots X_n]$$

be a family of polynomials in n variables and let

$$I = \sum \mathbb{Q}.P_i \subset \mathbb{Q}[X_1, \cdots x_n]$$

be the ideal they generate. We let

$$V_{P_1,\cdots,P_r}(\mathbb{Q}) = V_I(\mathbb{Q}) \subset \mathbb{Q}^n$$

be the subset of n-uples $\mathbf{x} \in \mathbb{Q}^n$ satisfying

$$P_1(\mathbf{x}) = \cdots = P_r(\mathbf{x}) = 0, \tag{5.1}$$

or equivalently

$$\forall P \in I,\ P(\mathbf{x}) = 0 \tag{5.2}$$

The set $V_I(\mathbb{Q})$ is the set of $\mathbb{Q}$-points of the algebraic variety defined by the equations (5.1) or (5.2).

Let R be a $\mathbb{Q}$ algebra; we can evaluate any polynomial $P(X) \in \mathbb{Q}[X_1, \cdots, X_n]$ at some R-valued n-uples $\mathbf{x} \in R^n$: if

$$P(X_1, \cdots X_n) = \sum_{k_1,\cdots,k_n}\sum a_{k_1,\cdots,k_n} X_1^{k_1} \cdots X_n^{k_n}$$

one has

$$P(\mathbf{x}) = \sum_{k_1,\cdots,k_n}\sum a_{k_1,\cdots,k_n} x_1^{k_1} \cdots x_n^{k_n} \in R.$$

We can then define the set of R-point of V_I as

$$V_I(R) = \{\mathbf{x} \in R^n,\ P_i(\mathbf{x}) = 0,\ i = 1, \cdots, r\}$$

Applying this to $R = \mathbb{A}$ or more generally $\mathbb{A}_V$ we obtain

$$V_I(\mathbb{A}_V) = \{\mathbf{x} \in \mathbb{A}_V^n,\ P_i(\mathbf{x}) = 0,\ i = 1, \cdots, r\} = \prod_{v \in V}{}' V_I(\mathbb{Q}_v)$$
$$= \{(\mathbf{x}_v)_v \in \mathbb{A}_V^n,\ \mathbf{x}_v \in V_I(\mathbb{Q}_v),\ \mathbf{x}_p \in \mathbb{Z}_p^n \text{ for a.e. } p \in V\}.$$

We have

$$V_I(\mathbb{Q}) = \mathbb{Q}^n \cap V_I(\mathbb{A}_V) \subset V_I(\mathbb{A}_V).$$

PROPOSITION 4.4. *For R any of the ring $\mathbb{Q}_v,\ \mathbb{A}_V,\ \mathbb{A}$, the sets $V_I(R)$ are closed subsets of R^n. In particular if K is a vector subspace of $\mathbb{Q}^n$ the R-submodule $K(R) \subset R^n$ generated by the elements of K is closed.*

PROOF. Indeed each function $\mathbf{x} \to P_i(\mathbf{x})$ is a continuous function on R^n for the corresponding topology. □

REMARK 5.1. Since $\mathbb{Q}^n$ is discrete in $\mathbb{A}^n$, $V_I(\mathbb{Q})$ is clearly discrete in $V_I(\mathbb{A})$. On the other hand neither the weak or the strong approximation theorems hold for V_I in general: there exist algebraic varieties such that $V_I(\mathbb{Q})$ is empty and $V_I(\mathbb{Q}_v)$ is non-empty for every v!