CHAPTER 1

Overview of Zhang's and Maynard's theorems

Let $\mathcal{P} = \{2, 3, 5, 7, 11, \cdots\}$ be the set of prime numbers; it is known at least since Euclide that $\mathcal{P}$ is infinite. In order to quantify how "dense" $\mathcal{P}$ might be, it is natural to introduce the prime counting function

$$\pi(x) = \sum_{p\leq x} 1 = \sum_{n\leq x} 1_{\mathcal{P}}(n).$$

Gauss made extensive numerical investigations on the growth of $\pi(X)$ and made the following conjecture

Conjecture (Gauss). *On has*

$$\pi(x) \simeq \ell i(x) \simeq \frac{x}{\log x}$$

where

$$\ell i(x) = \int_2^x \frac{dt}{\log t}$$

is called the integral logarithm.

Gauss prime number conjecture was eventually proven about 100 years later, independently and simultaneously by J. Hadamard and C. de la Vallée-Poussin:

Theorem. *Gauss's prime number conjecture holds. Equivalently, if we note*

$$\theta(x) := 1_{\mathcal{P}}(n) \log n,$$

one has

$$\Theta(x) = \sum_{n\leq x} \theta(n) = \sum_{p\leq x} \log p \simeq x.$$

In particular this theorem asserts that the set of prime numbers becomes less and less dense: the probability that an integer less than x and picked at random is $\sim 1/\log x \to 0,\ x \to \infty$.

In other terms, if we denote by p_n denote the n-th prime ($p_1 = 2,\ p_2 = 3, \cdots$), one has

$$p_n \sim n \log n$$

and the distance (the prime gap) between p_n and the next consecutive prime p_{n+1} satisfies

$$p_{n+1} - p_n \sim \log n$$

"on average". In this respect, the twin prime conjecture, made after numerical investigations is largely counter-intuitive:

CONJECTURE (Polignac). *There are infinitely many primes n such that*

$$p_{n+1} - p_n = 2.$$

In fact the twin primes conjecture has a more precise form

CONJECTURE. *Let*

$$\pi_2(x) = \sum_{\substack{p\leq x\\ p+2 \ \mathit{prime}}} 1$$

one has

$$\pi_2(x) \simeq 2C_2\frac{x}{(\log x)^2},\ C_2 := 2\prod_{p>2}(1-\frac{1}{(p-1)^2})$$

REMARK 1.1. This conjecture is in line with a naive probabilistic model asserting that the probability for an integer $n \leq x$ to be prime is $1/\log x$ and the probability for both n and $n+2$ being both primes should be $1/\log^2 x$.

One possible way to approach the twin prime conjecture is to investigate how small the gap $p_{n+1} - p_n$ can be; for this people considered the quantity

$$\liminf_n \frac{p_{n+1}-p_n}{\log p_n}.$$

and could prove that

$$\liminf_n \frac{p_{n+1}-p_n}{\log p_n} \leq c$$

for various values of $c < 1$. In 2005 a major breakthrough came came from the work of Goldston-Pintz-Yildirim who proved that

THEOREM 1.1 (GPY).

$$\liminf_n \frac{p_{n+1}-p_n}{\log p_n} = 0.$$

This breakthrough came from the invention of a new approach the problem of finding small gaps between primes.

In the spring of 2013, an unknown analytic number theorist, Y. Zhang, stunned the mathematical world by proving that

THEOREM 1.2 (Y. Zhang). *The quantity*

$$G_2 := \liminf_n p_{n+1} - p_n$$

is finite. More precisely, G_2 satisfies the bound

$$G_2 \leq 70.10^6.$$

The release of Zhang's paper generated considerable activity. In particular, a collaborative project, Polymath8, coordinated by Terence Tao was set up; its aim was, first to understand Zhang's work and whenever possible to improve the value of G_2.

In september of 2013, the polymath8 project reached a value $G_2 \leq 4680$ by improving Zhang's argument at several places.

However in October 2013, a postdoctoral assistant at Montreal, J. Maynard, made a second breakthrough by proving the bound

Theorem 1.3 (Maynard).

$$G_2 \leq 600.$$

This striking bound was obtained by a significantly simpler method; this is this method which we are going to expose in this course. Maynard's method allows for much stronger results non accessible to previous techniques: the existence of infinitely many l-uples of primes clustered in intervals of bounded length:

Theorem 1.4 (Maynard). *For any $l \geq 2$, the quantity*

$$G_l = \liminf_n p_{n+l} - p_n.$$

is finite.

Subsequently, Maynard joined the Polymath8 project and these bounds were further to improved (April 2014)

$$G_2 \leq 246$$

and

$$G_l = O(l \exp((4 - \frac{52}{283})m)).$$

1. The GPY method

The origin of these latest development comes from the work of GPY and its new method to handle efficiently the detection of prime gaps; this is based on wide generalization of the twin prime conjecture, the Hardy-Littlewood conjecture:

Given $k \geq 2$ and

$$\mathbf{h} = (h_1, \cdots h_k) \in \mathbb{Z}^k,\ h_1 < h_2 < \cdots < h_k$$

a strictly increasing k-tuple of integers. We consider the sequence of shifts

$$(n + h_1, \cdots, n + h_k)_{n \geq 1}.$$

The Hardy-Littlewood conjecture aims at predicting that the above vector has all of its entries primes for infinitely many integers n. For this conjecture to have a chance to hold, it is necessary that $\mathbf{h}$ satisfies some additional condition named *admissibility.* for instance suppose that $k = 2$ and $\mathbf{h} = (0, 1)$ then either n or $n + 1$ will always be prime therefore they cannot be simultaneously primes (unless $n = 2$).

DEFINITION 1.1. *A k-uple $\mathbf{h} = (h_1, \cdots h_k)$ is admissible of for any prime p*

$$\{h_i \,(\mathrm{mod}\, p),\ i = 1 \cdots k\} \neq \mathbb{Z}/p$$

The Hardy-Littlewood conjecture is the statement that this condition is also sufficient:

CONJECTURE (Hardy-Littlewood). *Let $\mathbf{h} \in \mathbb{Z}^k$ be admissible then*

$$\pi_{\mathbf{h}}(x) := \sum_{n \leq x} 1_{\mathcal{P}}(n + h_1) \cdots 1_{\mathcal{P}}(n + h_k) \simeq C_k \frac{x}{\log^k x}$$

with

$$C_k = \prod_p \frac{1 - h_p/p}{(1 - 1/p)^k}.$$

Equivalently

$$\sum_{n \leq x} \theta(n + h_1) \cdots \theta(n + h_k) \simeq C_k x.$$

The idea of GPY is the following: consider the sequence

$$(n + h_1, \cdots, n + h_k)_{n \geq 1}$$

for $\mathbf{h}$ admissible; suppose one can prove that this vector as at least two prime entries for infinitely many n (which is guaranteed by the HL conjecture) this implies that there are infinitely many prime gaps of size $\leq h_k - h_1$. This is the extra flexibility introduced by the extra parameter k which will eventually make this approach to work: let

$$\theta_{\mathbf{h}}(n) := \sum_{i=1}^{k} \theta(n + h_i);$$

the main observation is that if

$$\theta_{\mathbf{h}}(n) \geq \log(3n)$$

infinitely often, the sum $\theta_{\mathbf{h}}(n)$ contains at least 2 non-zero terms infinitely often.

Therefore, it woud be sufficient to evaluate the sum

$$\sum_{n \sim x} (\theta_{\mathbf{h}}(n) - \log(3n))$$

and show that it is positive infinitely often. This naive strategy has no chance to work since

$$\sum_{n \sim x} \theta_{\mathbf{h}}(n) \simeq kx, \ \sum_{n \leq x} \log(3x) \sim x \log(3x).$$

Instead GPY consider the weighted sum

$$\sum_{n \sim x} w(n)(\theta_{\mathbf{h}}(n) - \log(3n))$$

for $w(n)$ somenon-negative weights designed to "penalize" the integers n such that the product

$$P_{\mathbf{h}}(n) = \prod_i (n + h_i)$$

has many prime factors: the effect of these weights being that (assuming k is sufficiently large but absolutely bounded)

$$\sum_{n \leq x} w(n) \log(2n) \leq Q_{1,w} x(1 + o(1)) \tag{1.1}$$

while

$$\sum_{n \leq x} w(n)\theta_{\mathbf{h}}(n) \geq Q_{2,w} x(1 + o(1)) \tag{1.2}$$

for some constants $Q_{1,w},\ Q_{2,w} > 0$ such that

$$Q_{2,w}/Q_{1,w} > 1.$$

REMARK 1.2. Observe that the lower bound

$$Q_{2,w}/Q_{1,w} > l - 1$$

implies that $G_l < \infty$.

1.1. The GPY choice. The design of weights $w(n)$ penalizing integers with many prime factors (are therefore detecting almost prime numbers) belong to the theory of the *sieve* which is an art in itself and is to be discussed later. The GPY original choice was (roughly) of the following shape

$$w_{GPY}(n) = (\sum_{\substack{d|P_{\mathbf{h}}(n) \\ (d,\Delta_{\mathbf{h}})=1}} \mu(d) g(\frac{\log d}{\log z})^2$$

where

$$\Delta_{\mathbf{h}} = \prod_{j>i} (h_j - h_i)$$

is the discriminant[1] of $P_{\mathbf{h}}(X)$, $\mu(n)$ is the Moebius function, which is the multiplicative function defined for $n = \prod_{p|n} p^{\alpha_p}$ by

$$\mu(\prod_{p|n} p^{\alpha_p}) = \prod_{p|n} \mu(p^{\alpha_p}),$$

$$\mu(1) = 1,\ \mu(p) = -1,\ \mu(p^\alpha) = 0,\ \alpha \geq 2,$$

g is a suitable non-negative smooth function, compactly supported in $[0, 1]$ which will be optimized later and $z = x^\delta,\ \delta > 0$ is a truncation parameter which we will need to take sufficiently large to insure a good almost-prime

[1] this extra condition is natural for this problem, see below.

detection. The basic idea behind this choice (natural within Sieve methods) is that when z is small compared to n the sum

$$\lambda(n)=\sum_{d|n}\mu(d)g(\frac{\log d}{\log z})$$

will not have much time to oscillate and deviate much from the initial value of the sum $\mu(1)=1$ and will stay large in general while for z large compared to n, there will be many oscillations in this sum which will be therefore be small (remember that for $n\geq 1$,

$$\sum_{d|n}\mu(d)=0.)$$

The sum (1.1) is not too hard to evaluate and one has

$$\sum_{n\leq x}w_{GPY}(n)\log(3n)\simeq Q_1(g;k,\delta)x \tag{1.3}$$

where $Q_1(g,k,\delta)$ is a quadratic form in g.

The sum (1.2) constitute the hardest part: opening the square in the definition of g_{GPY}, one finds that this term equal

$$\sum_{\substack{d,d'\leq z^2\\(dd',\Delta_{\mathbf{h}})=1}}\cdots\sum_{i}\sum_{\substack{n\leq x\\ [d,d']|P_{\mathbf{h}}(n)}}\theta(n+h_i) \tag{1.4}$$

$$=\sum_{i}\sum_{\substack{q\leq z^2\\(dd',\Delta_{\mathbf{h}})=1}}(\sum_{[d,d']=q}\cdots)\sum_{\substack{h_i\leq n\leq x+h_i\\ P_{\mathbf{h}}(n-h_i)\equiv 0\,(\mathrm{mod}\,q)}}\theta(n)$$

where

$$\cdots=\mu(d)\mu(d')g(\frac{\log d}{\log z})g(\frac{\log d'}{\log z})$$

The innermost sums is a log-weighted count for the number of primes in the shifted sequence $n+h_i$ and in the union of arithmetic progressions modulo $q=[d,d']$, $a+q\mathbb{Z}$, for a defined by the polynomial congruence equation

$$P_{\mathbf{h}}(a-h_i)\equiv 0\,(\mathrm{mod}\,q).$$

Therefore we need first to be able to count primes of size x contained in arithmetic progressions of relatively large moduli (up to $q\approx x^{2\delta}$).

Observe that since q is coprime with $\Delta_{\mathbf{h}}$, the solutions of the above polynomial congruence equation are composed of

- the zero class $0\,(\mathrm{mod}\,q)$ and the corresponding sum is very small

$$\sum_{\substack{h_i\leq n\leq x+h_i\\ n\equiv 0\,(\mathrm{mod}\,q)}}\theta(n)\leq\log q,$$

- a number (bounded by $k^{\omega(q)}$) of congruences classes a which are coprime with q. We evaluate the corresponding sums

$$\sum_{\substack{h_i\leq n\leq x+h_i\\ n\equiv a\,(\mathrm{mod}\,q)}}\theta(n)$$

below.

2. Primes in arithmetic progressions to large moduli

THEOREM (Dirichlet). *Given $(a,q)=1$, the set*

$$\mathcal{P}\cap a+q\mathbb{Z}$$

is infinite.

REMARK 1.3. Observe that for $(a,q)\neq 1$, the set $\mathcal{P}\cap a+q\mathbb{Z}$ is finite.

This is a qualitative statement and we will need something more quantitative in several aspects. Let us recall that the number of arithmetic progressions of modulus q, $a+q\mathbb{Z}$ made of integers coprime with q (the only ones which could possibly contain infinitely many primes) equals the Euler totient function

$$|\{a\,(\mathrm{mod}\,q),\ (a,q)=1\}|=|(\mathbb{Z}/q)^\times|=\varphi(q)=q\prod_{p|q}(1-\frac{1}{p}),$$

We know from Dirichlet theorem that any such class contains infinitely many primes but there is a priori no reason why one class should contain more prmes than the other classes; this is indeed not the case a fact confirmed by more qualitative results: Dirichlet himself proved that

$$\sum_{p\equiv a\,(\mathrm{mod}\,q),p\leq x}\frac{\log p}{p}\simeq\frac{1}{\varphi(q)}(\sum_{p\leq x}\frac{\log p}{p})\simeq\frac{\log x}{\varphi(q)}$$

and subsquently Landau generalized the HdVP method to prove the following: let

$$\Theta(x;q,a)=\sum_{\substack{n\leq x\\ n\equiv a\,(\mathrm{mod}\,q)}}\theta(n)=\sum_{\substack{p\leq x\\ p\equiv a\,(\mathrm{mod}\,q)}}\log p$$

THEOREM (Landau). *Given $(a,q)=1$,*

$$\Theta(x;q,a)\simeq\frac{1}{\varphi(q)}x.$$

In other terms, there are asymptotically as many primes in any of the arithmetic progression $a\,(\mathrm{mod}\,q)$ for a prime to q.

From this result on can guess the expected main term for the sum (1.4) is asymptotic to

$$k\sum_{\substack{q\leq z^2\\ (q,\Delta_{\mathbf{h}})=1}}\frac{1}{\varphi(q)}\sum_{[d,d']=q}\cdots\times\Theta(x)\simeq Q_2(g;k,\delta)x. \tag{1.5}$$

Suppose that the step of replacing the sum (1.4) by the main term computed in this way can be justified for some value of δ, we would get

$$Q_2(g;k,\delta)/Q_1(g;k,\delta) \tag{1.6}$$

and would "just" have to choose g appropriately in order to maximize the above quotient (this is an optimization problem in analysis) and hope that its value is > 1.

We therefore need to evaluate the error term arising from making this approximatio and is essentially bounded by

$$\sum_i \sum_{\substack{q\leq z^2\\ (q,\Delta_{\mathbf{h}})=1}} \Big(\sum_{[d,d']=q}\cdots\Big) \sum_{\substack{(a,q)=1\\ P_{\mathbf{h}}(a-h_i)\equiv 0\,(\mathrm{mod}\,q)}} \Big|\Theta(x;q,a)-\frac{1}{\varphi(q)}\Theta(x)\Big|.$$

In the previous statement the dependency in q was not explicited but if it were, it would be very poor: the following is known

Theorem (Siegel-Walfisz). *For any $A \geq 0$*

$$\Theta(x;q,a)=\frac{1}{\varphi(q)}\Theta(x)+O_A\Big(\frac{\Theta(x)}{\log^A x}\Big).$$

In particular, if $q \leq (\log x)^{A-1}$,

$$x\log^{-A}x = o(\Theta(x;q,a)) = o\Big(\frac{1}{\varphi(q)}x\Big)$$

and one can count accurately primes less than x in an arithmetic progressions of sufficiently small modulus ($\leq (\log x)^{A-1}$) for any $A \geq 1$. Here we need to control the error term for the distribution of primes $\leq x$ in arithmetic progressions for much larger moduli, namely of size $x^{2\delta}$.

That this is indeed possible is a very deep

Conjecture (Generalized Riemann Hypothesis). *Given $(a,q)=1$, one has for any $x \geq 2$*

$$\Theta(x;q,a)=\frac{1}{\varphi(q)}\Theta(x)+O_\varepsilon(x^{1/2}\log^2 x).$$

In particular, one has for any $\varepsilon > 0$ and $x \geq 2$

$$\Theta(x;q,a)\simeq\frac{1}{\varphi(q)}\Theta(x)$$

as long as $q \leq x^{1/2-\varepsilon}$ and (1.5) would be valid for any $\delta < 1/4$.

Fortunately, for this specific problem (and many other), it is not necessary to be able to count primes in arithmetic progressions in large *individual* moduli q; we need only this count *on average* over $q \leq x^{2\delta}$. For this one has the following unconditional result (which match GRH):

Theorem (Bombieri-Vinogradov). *For any $\theta < 1/2$ and any $A \geq 0$*

$$\sum_{q\leq x^\theta}\max_{(a,q)=1}\Big|\Theta(x;q,a)-\frac{1}{\varphi(q)}\Theta(x)\Big| \ll \frac{x}{\log^A x}.$$

Consequently, one can prove rigorously (1.5) for any $\delta < 1/4$; unfortunately for $\delta = 1/4$ (so that $2\delta = 1/2$) it is possible to show that the quotient (1.6) is always ≤ 1 for k any fixed number; eventually by allowing k to grow slowly with x and by an additional device GPY were able to bypass the problem and to prove their main result.

On the other hand, GPY also proved that if $\delta > 1/4$, then for suitable g and sufficiently large k, one has

$$Q_2(g;k,\delta)/Q_1(g;k,\delta) > 1$$

therefore, if one could prove a version of the Bombieri-Vinogradov theorem for some $\theta > 1/2$ and therefore

THEOREM (GPY). *Assume that the Bombieri-Vinogradov Theorem holds with $1/2$ replaced by $1/2 + \eta$ for some fixed $\eta > 0$, then*

$$G_2 < \infty.$$

3. Zhang's contribution

The improvement on the Bombieri-Vinogradov theorem (passing the 1/2-threshold) was showed to be possible in some situations by Fouvry-Iwaniec and Bombieri-Friedlander-Iwaniec. The main new ingredient neede for this is a *smoothness assumption on the moduli q*

DEFINITION 1.2. *Given $y > 0$, an integer q is y-smooth if all its prime divisors are $\leq y$.*

In particular when y is small compared to q, q has many prime factors and admits a factorisation into products $q = q_1q_2$ where the size of of the two factors can be chosen flexibly (a smallest y implying a greater flexibility).

Because of this, Motohashi and Pintz modifies the GPY sieving argument so that the moduli q of the arithmetic progression become smooth and therefore so that the extension of BV become applicable (with some smoothness parameter $y = x^\sigma$. For this, it was necessary to change the weights $w_{GPY}(n) \leftrightarrow w_{MP}(n)$ and to recompute a new quotient

$$Q_2(g;k,\delta,\sigma)/Q_1(g;k,\delta,\sigma).$$

which could be proven > 1 for k large enough and $\delta > 1/4$. Unfortunately the works Fouvry-Iwaniec and Bombieri-Friedlander-Iwaniec were still not applicable.

Zhang's main achievement was to show that it was possible to pass the exponent 1/2 in the Bombieri-Vinogradov theorem therefore prove that $G_2 < \infty$. A stronger form of Zhang's original theorem is the following result

THEOREM (Zhang+Polymath8). *For any $\theta < 1/2 + 7/300$, there exist $\sigma > 0$ such that for any $A \geq 1$*

$$\max_{a \geq 1} \sum_{\substack{q \leq x^\theta,\ (a,q)=1 \\ q\ x^\sigma-smooth}} |\Theta(x;q,a) - \frac{1}{\varphi(q)}\Theta(x)| \ll_A \frac{x}{\log^A x}.$$

4. Maynard method

The main innovation of Maynard a new choice of the weights $w(n)$. Instead of considering a weight detecting whether the product

$$P_{\mathbf{h}}(n) = \prod_i (n + h_i)$$

has few or many prime factors, Maynard introduced a weight for each factor of the polynomial; a simplified version of Maynard weight is the following

$$w_M(n) = \prod_{i=1}^{k} (\sum_{\substack{d \mid n+h_i \\ (d,\Delta_{\mathbf{h}})=1}} \mu(d) g(\frac{\log d}{\log z}))^2.$$

With these new weights one can carry out the same analysis of the main terms and reach a quotient

$$Q_{2,M}(g;k,\delta)/Q_{1,M}(g;k,\delta).$$

The main feature of these weights is the following:

THEOREM (Maynard). *For and $\delta > 0$ and any $\ell \geq 1$ one has, for k sufficiently large*

$$Q_{2,M}(g;k,\delta)/Q_{1,M}(g;k,\delta) > \ell - 1$$

for some choice of g.

By the previous discussion this implies Maynard's theorem on clusters of primes in bounded intervals: for any $\ell \geq 1$

$$\liminf p_{n+\ell} - p_n < \infty.$$