CHAPTER 2

Primes in arithmetic progressions to small moduli

1. The prime number Theorem and Riemann zeta function

In this section we review the Hadamard-de la Vallée-Poussin prime number theorem:

Theorem (Hadamard-de la Vallée-Poussin). *Let*

$$\Theta(x) = \sum_{p\leq x} \log p = \sum_{n\leq x} \theta(n) \simeq x.$$

An equivalent way to formulate the theorem (and a way to prove it) is by introducing the von Mangolt functions

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha,\ \alpha \geq 1 \\ 0 & \text{else} \end{cases}$$

and its summatory function

$$\psi(x) = \sum_{n\leq x} \Lambda(n).$$

Since

$$\Theta(x) = \psi(x) + O(x^{1/2}\log x)$$

the previous version of the PNT follows from

Theorem (Hadamard-de la Vallée-Poussin). *There exists $C > 0$ such that*

$$\psi(x) = x + O(x\exp(-C\sqrt{\log x})).$$

This version of te PNT is deduced from the study of the analytic properties of Riemann's zeta function defined by

$$\zeta(s) = \sum_{n\geq 1} n^{-s} = \prod_p (1-\frac{1}{p^s})^{-1},\ \Re s > 1$$

and extended to $\mathbb{C}$ by analytic continuation. More precisely the von Mangolt function $\Lambda(n)$ occur in the coefficients of the logarithmic derivative

$$-\frac{\zeta'}{\zeta}(s) = \sum_p \frac{(1-1/p^s)'}{(1-1/p^s)} = \sum_p \frac{\log p}{p^s}(1-\frac{1}{p^s})^{-1} = \sum_{n\geq 1} \Lambda(n)n^{-s}.$$

1.1. Mellin transform. The passage from the summatory function

$$\psi(x)=\sum_{n\leq x}\Lambda(n)$$

to its associated Dirichlet L-series

$$-\frac{\zeta'}{\zeta}(s)=\sum_{n\geq 1}\Lambda(n)n^{-s}$$

is made via Mellin transform:

Let $\varphi\in\mathcal{S}^{\infty}(\mathbb{R})$ be a function in the Schwartz class (with rapid decay at ∞ along with all its derivatives); its Mellin transform is the complex valued function defined for $\Re s>0$ by

$$\tilde{\varphi}(s)=\int_0^{\infty}\varphi(x)x^s\mathrm{d}^{\times}x.$$

Since φ rapid decay at ∞, $\tilde{\varphi}$ is holomorphic it the half plane $\{\Re s>0\}\subset\mathbb{C}$. Moreover integrating repeatedly by parts, one has

$$\tilde{\varphi}(s)=\frac{(-1)^k}{s(s+1)\cdots(s+k-1)}\tilde{\varphi^{(k)}}(s+k)$$

which shows (from applying the previous reasonning to $\varphi^{(k)}$ which is still in the Schwartz class) that $\tilde{\varphi}(s)$ extend meromorphically to $\mathbb{C}$ with at most simple poles at $-\mathbb{N}=\{0,-1,-2,\cdots\}$. In addition, the same argument shows that if φ is zero in a neighborhood of 0 then $\tilde{\varphi}$ is holomorphic on $\mathbb{C}$. Finally the above bound show that one has for any $k\geq 0$ and $|t|\geq 1$,

$$\tilde{\varphi}(\sigma+it)\ll_k|t|^{-k} \tag{2.1}$$

uniformly σ in a bounded interval of $\mathbb{R}$ therefore $\tilde{\varphi}$ is rapidly decreasing in vertical strips.

One can also reconstruct φ from $\tilde{\varphi}$ via the Fourier inversion formula: for any $\sigma>0$

$$\varphi(x)=\frac{1}{2\pi i}\int_{(\sigma)}\tilde{\varphi}(s)x^{-s}ds.$$

REMARK 2.1. If φ vanishes near 0 this identity is valid for any σ.

This implies that for any $\sigma>1$

$$\sum_n\Lambda(n)\varphi(n)=\sum_n\Lambda(n)\frac{1}{2\pi i}\int_{(\sigma)}\tilde{\varphi}(s)n^{-s}ds=\frac{1}{2\pi i}\int_{(\sigma)}\tilde{\varphi}(s)(-\frac{\zeta'(s)}{\zeta(s)})ds.$$

We have therefore expressed the sum

$$\sum_n\Lambda(n)\varphi(n)$$

as a complex integral involving the logarithmic derivative of the Riemann zeta function. Using this formula, we will deduce non-trivial estimates on the left-hand side sum from non-trivial analytic properties of ζ.

1.2. The analytic continuation and function equation for ζ. Our first step is to extend ζ analytically to $\mathbb{C}$.

Let
$$\Gamma(s) = \int_0^\infty \exp(-x)x^s \mathrm{d}^\times x,\ \Re s > 0$$
be the Mellin transform of $x \to \exp(-x)$. This function is also called Euler's Γ function and we recall that it interpolates the factorial function at positive integers: for $n \geq 0$
$$\Gamma(n+1) = n!.$$
This function is holomorphic in $\{\Re s > 0\}$ and extend analytically to $\mathbb{C}$ with simple poles at $-\mathbb{N}$; it can also be shown that $\Gamma(s)$ does not vanish on $\mathbb{C}$ (see below). The following result is a special case of a more general one which we prove below:

Theorem 2.1. *Let*
$$\zeta_\infty(s) := \pi^{-s/2}\Gamma(s/2)$$
and
$$\Lambda(s) := \zeta_\infty(s)\zeta(s);$$
then $\Lambda(s)$ extends meromorphically to $\mathbb{C}$ with simple poles at $s = 0, 1$ and satisfies the functional equation
$$\Lambda(s) = \Lambda(1-s);$$
consequently $-\frac{\zeta'}{\zeta}(s)$ admit meromorphic continuation to $\mathbb{C}$ with poles located at the zeros and poles of
$$\Lambda(s)/\zeta_\infty(s).$$

1.3. Basic localization of the zero of the Riemann zeta function. Because of the last statement we will need to locate the zeros of $\Lambda(s)$:

- $\Lambda(s)$ does not vanish for $\Re s > 1$
- $\Lambda(s)$ does not vanish for $\Re s < 0$
- All the zero of Λ have real part in $[0, 1]$
- The zeros of $\zeta(s)$ are either the negative even integers $-2\mathbb{N}_{\geq 1}$ or the zeros of Λ.

Definition 2.1. *The set of negative even integers $-2\mathbb{N}_{\geq 1}$ is called the set of trivial zeros of ζ. The other zeros (the zeros of Λ) are called non-trivial. We will often use the notation*
$$\rho = \sigma + i\gamma$$
to denote a non-trivial zero. The vertical strip
$$\{s \in \mathbb{C}, \Re s \in [0, 1]\}$$
is called the critical strip and the vertical line $\{s \in \mathbb{C}, \Re s = 1/2\}$ is called the vertical line.

The first statement follows from the identity

$$\Lambda(s) = \zeta_\infty(s) \prod_p (1 - p^{-s})^{-1}, \ \Re s > 1$$

and the fact that $\Gamma(s)$ never vanishes on $\mathbb{C}$. The second statement is the functional equation. The third one is evident from the first two (and the knowledge of the pole and the absence of zeros if Γ).

1.4. The Explicit formula. We will establish later the following "Explicit formula" which relates precisely the problem of counting primes to the location of the zeros of ζ The relation between the :

THEOREM (Explicit formula). *Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_{>0})$, and $\check{\varphi}(x) := x^{-1}\varphi(x^{-1})$, one has the identity*

$$\sum_n \Lambda(n)\varphi(n) + \sum_n \Lambda(n)\check{\varphi}(n)$$

$$= \int_0^\infty \varphi(x)dx - \sum_\rho \tilde{\varphi}(\rho) + \frac{1}{2\pi i} \int\limits_{(1/2)} [\frac{\zeta'_\infty(s)}{\zeta_\infty(s)} + \frac{\zeta'_\infty(1-s)}{\zeta_\infty(1-s)}]\tilde{\varphi}(s)ds + \int_0^\infty \varphi(x)\mathrm{d}^\times x.$$

Here the (absolutely converging) ρ-sum is along the non-trivial zeros of ζ counted with multiplicity.

REMARK 2.1. Implicit in the statement of this theorem is the convergence in absolute value of the above infinite sums and intergrals. This will be justified later.

REMARK 2.2. If $\mathrm{supp}(\varphi) \subset]1, +\infty[$ the formula simplifies to

$$\sum_n \Lambda(n)\varphi(n) = \int_0^\infty \varphi(x)dx$$
$$- \sum_\rho \tilde{\varphi}(\rho) + \frac{1}{2\pi i} \int\limits_{(1/2)} [\frac{\zeta'_\infty(s)}{\zeta_\infty(s)} + \frac{\zeta'_\infty(1-s)}{\zeta_\infty(1-s)}]\tilde{\varphi}(s)ds + \int_0^\infty \varphi(x)\mathrm{d}^\times x.$$

Let us now replace φ by $\varphi(x/X)$ with $\varphi \in \mathcal{C}_c^\infty(]1, +\infty[)$ fixed; we have for X large enough

$$\sum_n \Lambda(n)\varphi(n/X) = X \int_0^\infty \varphi(x)dx$$
$$- X^\rho \sum_\rho \tilde{\varphi}(\rho) + \frac{1}{2\pi i} \int\limits_{(1/2)} [\frac{\zeta'_\infty(s)}{\zeta_\infty(s)} + \frac{\zeta'_\infty(1-s)}{\zeta_\infty(1-s)}]\tilde{\varphi}(s)X^s ds + \int_0^\infty \varphi(x)\mathrm{d}^\times x$$
$$= X \int_0^\infty \varphi(x)dx + O(\sum_\rho X^\sigma |\tilde{\varphi}(\rho))| + O_\varphi(X^{1/2})$$

which amounts to count primes in the the range $p \approx X$.

PROOF. We give a proof ignoring anyconvergence issues

$$\frac{1}{2\pi i}\int_{(2)}\frac{-\Lambda'(s)}{\Lambda(s)}\tilde{\varphi}(s)ds = \frac{1}{2\pi i}\int_{(2)}\frac{-\zeta'(s)}{\zeta(s)}\tilde{\varphi}(s)ds - \frac{1}{2\pi i}\int_{(2)}\frac{\zeta'_\infty(s)}{\zeta_\infty(s)}\tilde{\varphi}(s)ds$$

we then shift the contour to $\Re s = -1/2$ hitting poles at $s=1$, $s=0$ and at the zeros of $\Lambda(s)$:

$$\frac{1}{2\pi i}\int_{(2)}\frac{-\Lambda'(s)}{\Lambda(s)}\tilde{\varphi}(s)ds = \frac{1}{2\pi i}\int_{(-1/2)}\frac{-\Lambda'(s)}{\Lambda(s)}\tilde{\varphi}(s)ds + \tilde{\varphi}(1) + \tilde{\varphi}(0) - \sum_\rho \tilde{\varphi}(\rho)$$

and by the functional equation

$$\frac{-\Lambda'(s)}{\Lambda(s)} = -\frac{-\Lambda'(s)}{\Lambda(s)}$$

$$\frac{1}{2\pi i}\int_{(-1/2)}\frac{-\Lambda'(s)}{\Lambda(s)}\tilde{\varphi}(s)ds = -\frac{1}{2\pi i}\int_{(-1/2)}\frac{-\Lambda'(1-s)}{\Lambda(1-s)}\tilde{\varphi}(s)ds$$

$$= -\frac{1}{2\pi i}\int_{(3/2)}\frac{-\Lambda'(s)}{\Lambda(s)}\tilde{\varphi}(1-s)ds$$

□

1.4.1. *Convergence issues.* The proof provided here is not at all rigorous: we need to justify the convergence of the various integral involved and of the validity of the contour shift from the vertical line $\Re s = 2$ to the vertical line $\Re s = -1/2$. For this one require three ingredients:

The first one is (2.1), ie. the rapid decay of $\tilde{\varphi}(s)$ for s varying in vertical strips: uniformly for $\Re s \in [-10, 10]$ on has for any $k \geq 0$

$$\tilde{\varphi}(s) \ll_k (1+|s|)^{-k}.$$

The second one is the slow growth of the logarithmic derivative $\frac{\zeta_\infty}{\zeta_\infty}(s)$ as $s \to \infty$ and $\Re s \in [-10, 10]$: this is a consequence of one part of the Stirling's formula for the Gamma function (see below): for $|\Im s| \geq 1$ and $\Re s \in [-10, 10]$, one has

$$\frac{\Gamma'}{\Gamma}(s) \ll \log|s|.$$

The third and deepest ingredient is the following counting result for the non-trivial zeros of $\zeta(s)$: for $T \geq 0$ let

$$\mathcal{Z}(T) = \{\rho = \sigma + i\gamma,\ \Lambda(\rho) = 0,\ \sigma \in [0,1],\ |\gamma| \leq T\}$$

be the multiset (each zero is counted with its multiplicity) of zeros of $\Lambda(s)$ whose imaginary part is less than T and let

$$N(T) = |\mathcal{Z}(T)|$$

be the cardinality of this multiset (the sum of the multiplicities of the zeros).

THEOREM. *For $T \geq 0$, one has*

$$N(T+1) - N(T) = |\{\rho,\ \Lambda(\rho) = 0,\ |\gamma| \in]T, T+1]\}| \ll \log(2+T).$$

EXERCISE 2.1. *Assuming the previous results, prove rigorously the explicit formula.*

As said before the zeros of $\Lambda(s)$ are all contained in the critical strip and this should be considered evident information. The proof of the PNT comes from the possibility to locate zeros more precisely.

The following statement

THEOREM. *The function $\Lambda(s)$ has no zeros on the border line $\{s,\ \Re s = 1\}$.*

is sufficient to imply the PNT in the form

$$\psi(x) = x(1 + o(1)).$$

With more informations one can improve the error term: Hadamard and de la Vallée-Poussin proved that

THEOREM (H-dVP). *There exists a constant $C > 0$ such that $\Lambda(s)$ has no zeros in the domain*

$$\{s = \sigma + it,\ \sigma \geq 1 - C/\log(2 + |t|)\}.$$

EXERCISE 2.2. *Use the explicit formula to prove that this zero-free region implies that for any φ smooth compactly supported one has*

$$\sum_n \Lambda(n)\varphi(\frac{n}{x}) = x\int_0^\infty \varphi(u)du + O_\varphi(x\exp(-c\sqrt{\log x})).$$

Let us recall that the set of zeros of $\zeta(s)$ is invariant under the symmetry $s \leftrightarrow 1-s$ the best possible location for zeros one could imagine is

CONJECTURE (Riemann Hypothesis (RH)). *The zeros of $\Lambda(s)$ are located on the critical line $\{s,\ \Re s = 1/2\}$.*

EXERCISE 2.3. *Prove that the RH implies that*

$$\sum_n \Lambda(n)\varphi(\frac{n}{x}) = x\int_0^\infty \varphi(u)du + O_\varphi(x^{1/2}\log x).$$

2. Prime in arithmetic progressions

We now discuss the proof of the following

THEOREM (Dirichlet). *Given $(a, q) = 1$, the set*

$$\mathcal{P} \cap (a + q\mathbb{Z})$$

is infinite.

A for the prime this theorem can be given a more quantitative form via summatory functions: one may either form

$$\pi(x;q,a) = \sum_{\substack{p\leq x \\ p\equiv a\,(\mathrm{mod}\,q)}} 1$$

or the θ-summatory function in arithmetic progressions

$$\theta(x;q,a) = \sum_{\substack{p\leq x \\ p\equiv a\,(\mathrm{mod}\,q)}} \log p$$

or the von Mangolt summatory function

$$\psi(x;q,a) = \sum_{\substack{n\leq x \\ n\equiv a\,(\mathrm{mod}\,q)}} \Lambda(n).$$

One has the following three equivalent formulation of the prime number theorem in arithmetic progressions

Theorem (Landau). *Given $(a,q)=1$, as $x\to\infty$ one has*

$$\pi(x;q,a)\log x \simeq \Theta(x;q,a) \simeq \psi(x;q,a) \simeq \frac{1}{\varphi(q)}x.$$

Let us recall that $\varphi(q)$ is Euler's totient function

$$\varphi(q) = |(\mathbb{Z}/q)^{\times}| = q\prod_{p|q}(1-\frac{1}{p};$$

This theorem states that the reduction modulo q map

$$\mathrm{red}_q : p\in\mathcal{P}^{(q)} := \{p\in\mathcal{P},\ (p,q)=1\} \mapsto p\,(\mathrm{mod}\,q)\in(\mathbb{Z}/q)^{\times}$$

retricted to the primes less than x has its fibers all about the same size or in other terms, that the probability that a random large prime reduce modulo q to a given congruence class is $1/\varphi(q)$.

The following theorem make the dependency in x and q of these

Theorem (Siegel-Walfisz). *For any $A\geq 0$, one has*

$$\theta(x;q,a) = \frac{1}{\varphi(q)}x + O_A(x\log^{-A}x)$$

and equivalently

$$\psi(x;q,a) = \frac{1}{\varphi(q)}x + O_A(x\log^{-A}x).$$

In particular, if $q\leq(\log x)^{A-1}$,

$$x\log^{-A+1}x = o(\psi(x;q,a)) = o(\frac{1}{\varphi(q)}x)$$

and one can count accurately primes less than x in an arithmetic progressions of sufficiently small modulus ($\leq(\log x)^{A-1}$).

One expects to be able to do much better

Conjecture (Generalized Riemann Hypothesis). *Given* $(a, q) = 1$, *one has for any* $x \geq 2$

$$\psi(x; q, a) = \frac{1}{\varphi(q)} x + O_\varepsilon(x^{1/2} \log^2 x).$$

In particular, for any $\varepsilon > 0$*one has*

$$\psi(x; q, a) \simeq \frac{1}{\varphi(q)} x$$

unformly for $q \leq x^{1/2-\varepsilon}$.

3. Dirichlet L-functions

3.1. Dirichlet characters. We want to count primes which satisfy the additional congruence condition

$$p \equiv a \,(\mathrm{mod}\, q)$$

for $(a, q) = 1$ (recall that if $(a, q) \neq 1$ there are only the prime dividing (a, q)).

The traditional way to approach Dirichlet's theorem as well as its more precise forms is through the L-functions associated to Dirichlet characters: let us recall that

$$\mathbb{C}^{(\mathbb{Z}/q)^\times} = \{f : (\mathbb{Z}/q)^\times \to \mathbb{C}\}$$

the space of complex value functions on the group $(\mathbb{Z}/q)^\times$ is a complex vector space of dimension

$$|(\mathbb{Z}/q)^\times| = \varphi(q) = q \prod_{p|q} (1 - \frac{1}{p})$$

has two natural bases: the obvious "Dirac" base

$$\{\delta_{a\,(\mathrm{mod}\, q)},\ a \in (\mathbb{Z}/q)^\times\}$$

and the base of Dirichlet characters modulo q, ie. the group of group homomorphisms from $(\mathbb{Z}/q)^\times$ into $\mathbb{C}^\times$

$$\widehat{(\mathbb{Z}/q)^\times} = \mathrm{Hom}_\mathbb{Z}((\mathbb{Z}/q)^\times, \mathbb{C}^\times).$$

While the first base is natural to detect congruences conditions like

$$n \equiv a \,(\mathrm{mod}\, q),$$

the second base $\widehat{(\mathbb{Z}/q)^\times}$ is equality natural as it form an eigenbasis of functions on $(\mathbb{Z}/q)^\times$ (normalized by the condition $\chi(1) = 1$) for the natural action of $(\mathbb{Z}/q)^\times$ on $\mathbb{C}^{(\mathbb{Z}/q)^\times}$ induced by translations: for any $t, a \in (\mathbb{Z}/q)^\times$ one has $\chi(ta) = \chi(t)\chi(a)$. As such the group of characters captures the group structure on $(\mathbb{Z}/q)^\times$; both bases are orthogonal relative to the translation-invariant inner product

$$\langle f, g\rangle = \frac{1}{\varphi(q)} \sum_{a \,(\mathrm{mod}\, q)} \overline{f}(a)g(a)$$

and the passage from on base to the other is precisely "Fourier theory" on the abelian group $(\mathbb{Z}/q)^\times$:

$$\delta_{a\,(\mathrm{mod}\, q)} = \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)\chi,$$

or in other terms, one has

– Orthogonality of characters for any $(ab, q) = 1$

$$\frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)\chi(b) = \delta_{a\equiv b\,(\mathrm{mod}\, q)},$$

–

$$\frac{1}{\varphi(q)} \sum_{a\,(\mathrm{mod}\, q)} \overline{\chi}(a)\chi'(a) = \delta_{\chi=\chi'}.$$

In particular we have

$$\psi(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi\,(\mathrm{mod}\, q)} \overline{\chi}(a)\psi(x; \chi)$$

where

$$\psi(x; \chi) := \sum_{n\leq x} \Lambda(n)\chi(n).$$

The contribution of $\psi(x; q, a)$ to the trivial character $\chi = \chi_0 \equiv 1$ equals

$$\frac{1}{\varphi(q)}\psi(x; \chi_0) = \frac{1}{\varphi(q)} \sum_{\substack{n\leq x\\(n,q)=1}} \Lambda(n) = \frac{1}{\varphi(q)}(x+O(x \exp -(c(\log x)^{1/2})+O(\log q)).$$

It remains to show that for any $\chi \neq \chi_0$

$$\psi(x; \chi) = O(x \log^{-A} x) \tag{2.2}$$

for any $A \geq 0$.

This latter sum is the summatory function associated to the logarithmic derivative of the Dirichlet L-function

$$L(\chi, s) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1};$$

One has

$$-\frac{L'(\chi, s)}{L(\chi, s)} = \sum_{n\geq 1} \frac{\chi(n)\Lambda(n)}{n^s}.$$

and the estimate (2.2) will follow from the analytic properties of $L(\chi, s)$

3.2. Analytic properties of Dirichlet L-functions. As for the PNT, the proof of Landau's theorem builds on the analytic properties of Dirichlet L-functions

$$L(\chi,s)=\sum_n\frac{\chi(n)}{n^s}=\prod_p(1-\frac{\chi(p)}{p^s})^{-1},\ \Re s>1.$$

We will show that $L(\chi,s)$ extend analytically to $\mathbb{C}$ and satisfies a form of functional equation. To describe the later, it is better to make a small reduction first and to introduce the notion of

3.2.1. *Primitive characters.* Let $q'|q$ be a divisor; we have a natural surjective reduction modulo q' map

$$\mathrm{red}_{q'}:(\mathbb{Z}/q)^\times\mapsto(\mathbb{Z}/q')^\times$$

and consequently a natural injective map on characters

$$\mathrm{red}_{q'}^*:\widehat{(\mathbb{Z}/q')^\times}\hookrightarrow\widehat{(\mathbb{Z}/q)^\times}$$

given by

$$\mathrm{red}_{q'}^*:\chi'\,(\mathrm{mod}\,q')\mapsto a\,(\mathrm{mod}\,q)\to\chi'(\mathrm{red}_{q'}(a)).$$

Definition 2.2. *A character* $(\mathrm{mod}\,q)$ *is primitive if the only pair* (q',χ') *such that* $\mathrm{red}_{q'}^*(\chi')=\chi$ *is the trivial pair* (q,χ).

For any character $\chi\,(\mathrm{mod}\,q)$*, there is a unique primitive character* $\chi^\star(\,(\mathrm{mod}\,q)^\star)$ *with* $q^\star|q$ *such that* $\mathrm{red}_{q^\star}^*(\chi^\star)=\chi$*. The modulus* $q^\star$ *is called the conductor of* χ*.*

Remark 2.3. Equivalently, a character is primitive if χ is non trivial when restricted to any subgroup of the shape

$$\ker(\mathrm{red}_{q'})=\{x\in(\mathbb{Z}/q)^\times,\ x\equiv 1\,(\mathrm{mod}\,q)'\}$$

for any $q'|q$ and $q'<q$.

Let $L(\chi,s)$ be the Dirichlet L-function associated to some character χ with associated primitive character $\chi^\star$, we have

$$L(\chi,s)=\sum_{(n,q)=1}\frac{\chi^\star(n)}{n^s}=\prod_{p|q}(1-\frac{\chi^\star(p)}{p^s})L(\chi^\star,s).\tag{2.3}$$

In particular the study of $L(\chi,s)$ reduces to that of $L(\chi^*,s)$ when χ^* is a primitive character.

Remark 2.4. When the underlying primitive character is the trivial character (of modulus 1) χ_0 one has

$$L(\chi_0,s)=\zeta(s)$$

Theorem 2.2. *let* $\chi\,(\mathrm{mod}\,q)$ *be a primitive Dirichlet character and let*

$$L_\infty(\chi,s)=\begin{cases}\pi^{-s/2}\Gamma(s/2) & if\ \chi\ is\ even\\ \pi^{-(s+1)/2}\Gamma(\frac{s+1}{2}), & if\ \chi\ is\ odd.\end{cases}$$

and

$$\Lambda(\chi, s) := q^{s/2} L_\infty(\chi, s) L(\chi, s);$$

then $\Lambda(\chi, s)$ *extends meromorphically to* $\mathbb{C}$ *with a most two simple poles at* $s = 0, 1$*; the later occur if only if* $\chi = 1$ *(otherwise* $\Lambda(\chi, s)$ *is holomorphic); moreover* $\Lambda(\chi, s)$ *is a function of order at most* 1 *(outside of its poles if any); moreover it satisfies*

$$\Lambda(\chi, s) = \varepsilon_\chi \Lambda(\overline{\chi}, 1 - s) \tag{2.4}$$

where $|\varepsilon_\chi| = 1$.

Proof. Let $f \in \mathcal{S}(\mathbb{R})$ of the same parity as χ. For $\Re es > 1$ (since $|\chi(n)| \leq 1$)

$$\widetilde{f}(s) L(\chi, s) = \sum_{n \geq 1} \chi(n) \int_{\mathbb{R}_{>0}} f(x) (\frac{x}{n})^s \mathrm{d}^\times x = \int_{\mathbb{R}>0} (\sum_{n \geq 1} \chi(n) f(nx)) x^s \mathrm{d}^\times x.$$

Let

$$\theta(f, \chi; x) = \sum_{n \in \mathbb{Z}} \chi(n) f(nx).$$

This is a smooth even function of x satisfying for any $A \geq 1$

$$\theta(f, \chi; x) - \chi(0) f(0) = O(x^{-A}),\ x \to \infty, \tag{2.5}$$

$$\theta(f, \chi; x) = O(x^{-1}),\ x \to 0, \tag{2.6}$$

and we have

$$\sum_{n \geq 1} \chi(n) f(nx) = \frac{1}{2} [\theta(f, \chi; x) - \chi(0) f(0)].$$

We split the integral into two parts

$$\widetilde{f}(s) L(\chi, s) = \int_{\mathbb{R}_{>0}} \cdots = \int_0^1 \cdots + \int_1^\infty \cdots$$

By (2.5) the second integral

$$\frac{1}{2} \int_1^\infty [\theta(f, \chi; x) - \chi(0) f(0)] x^s \mathrm{d}^\times x$$

is holomorphic on $\mathbb{C}$ while the first is holomorphic for $\Re s > 1$ and potentialy singular at $s = 1$.

In the first integral, we make the change of variable $x \leftrightarrow 1/x$, we have

$$\int_0^1 \cdots = \frac{1}{2} \int_1^\infty [\theta(f, \chi; 1/x) - \chi(0) f(0)] x^{-s} \mathrm{d}^\times x.$$

We would like to apply Poisson summation formula: since $n \mapsto \chi(n)$ depends only on $n \,(\mathrm{mod}\, q)$ we have

$$\sum_{n \in \mathbb{Z}} \chi(n) f(n/x) = \sum_{a(q)} \chi(a) \sum_{m \in \mathbb{Z}} f(\frac{a + qm}{x}) = \sum_{a(q)} \chi(a) \sum_{m \in \mathbb{Z}} f(\frac{q}{x}(m + \frac{a}{q})).$$

By the Poisson summation formula we have

$$\sum_{m\in\mathbb{Z}} f(\frac{q}{x}(m+\frac{a}{q})) = \sum_{m\in\mathbb{Z}} \widehat{f_{\frac{q}{x}}}(m)e(-\frac{am}{q}) = \frac{x}{q}\sum_{m\in\mathbb{Z}} \widehat{f}(\frac{x}{q}m)e(-\frac{am}{q})$$

and

$$\sum_{n\in\mathbb{Z}}\chi(n)f(n/x) = \frac{x}{q}\sum_{m\in\mathbb{Z}}\widehat{f}(\frac{x}{q}m)\sum_{a(q)}\chi(a)e(-\frac{am}{q}) = \frac{x}{q}\sum_{m\in\mathbb{Z}}\widehat{f}(\frac{x}{q}m)\tau_\chi(-m)$$

where

$$\tau_\chi(m) = \sum_{a(q)}\chi(a)e(m\frac{a}{q})$$

is the Fourier transform (relative to the additive character $e(\frac{\cdot}{q})$ on $\mathbb{Z}/q$) of the function on $\chi : \mathbb{Z}/q \to \mathbb{C}$ (extended by 0 outside $(\mathbb{Z}/q)^\times$).

Since χ is primitive (see §3.2.2), one has

$$\tau_\chi(-m) = \chi(-1)\overline{\chi}(m)\tau_\chi$$

and

$$|\tau(\chi)| = q^{1/2}$$

and therefore

$$\theta(f,\chi;1/x) = \frac{x}{q}\chi(-1)\tau_\chi\sum_{m\in\mathbb{Z}}\widehat{f}(\frac{x}{q}m)\overline{\chi}(m) = \frac{x}{q}\chi(-1)\tau_\chi\theta(\widehat{f},\overline{\chi},\frac{x}{q}).$$

We eventually obtain

$$\int_0^1 \cdots = \frac{1}{2}\int_1^\infty \frac{\chi(-1)\tau_\chi}{q}\theta(\widehat{f},\overline{\chi},\frac{x}{q})x^{1-s} - \chi(0)f(0)x^{-s}\mathrm{d}^\times x$$

Writing

$$\theta(\widehat{f},\overline{\chi},\frac{x}{q}) = \theta(\widehat{f},\overline{\chi},\frac{x}{q}) - \overline{\chi}(0)\hat{f}(0) + \overline{\chi}(0)\hat{f}(0)$$

we decompose $\int_0^1 \dots$ as a sum of two terms:

$$\frac{1}{2}\int_1^\infty \frac{\chi(-1)\tau_\chi}{q}[\theta(\widehat{f},\overline{\chi},\frac{x}{q}) - \overline{\chi}(0)\hat{f}(0)]x^{1-s}\mathrm{d}^\times x$$

and

$$\frac{1}{2}\int_1^\infty [\overline{\chi}(0)\hat{f}(0)x^{1-s} - \chi(0)f(0)x^{-s}]\mathrm{d}^\times x = \frac{1}{2}[\overline{\chi}(0)\hat{f}(0)\frac{1}{s-1} - \chi(0)f(0)\frac{1}{s}]$$

By (2.5), the series $\theta(\widehat{f},\overline{\chi},\frac{x}{q}) - \chi(0)\widehat{f}(0)$ defines a rapidly decreasing function as $x \to +\infty$, the first x-integral is uniformly converging for s in compact subsets of $\mathbb{C}$ and therefore defines an holomorphic function on $\mathbb{C}$.

We have obtained that

$$\begin{aligned}2\widetilde{f}(s)L(\chi,s) =&\overline{\chi}(0)\hat{f}(0)\frac{1}{s-1}-\chi(0)f(0)\frac{1}{s}\\ &+\int_1^\infty[\theta(f,\chi;x)-\chi(0)f(0)]x^s\mathrm{d}^\times x\\ &+\int_1^\infty\frac{\chi(-1)\tau_\chi}{q}[\theta(\widehat{f},\overline{\chi},\frac{x}{q})-\overline{\chi}(0)\hat{f}(0)]x^{1-s}\mathrm{d}^\times x\end{aligned}$$

extends meromorphically to $\mathbb{C}$ and is holomorphic unless χ is trivial in which case it has at most two additional simple poles at $s=0,1$.

In the previous equality let us, replace f by $\widehat{f}$, χ by $\overline{\chi}$, s by $1-s$ and make the changes of variables $x\leftrightarrow x/q$ and $x/q\leftrightarrow x$ in the first and second integrals: we obtain

$$\begin{aligned}2\widetilde{\widehat{f}}(1-s)L(\overline{\chi},1-s) =&-\chi(0)\widehat{\widehat{f}}(0)\frac{1}{s}-\overline{\chi}(0)\widehat{f}(0)\frac{1}{1-s}\\ &+q^{s-1}\int_1^\infty[\theta(f,\chi;\frac{x}{q})-\overline{\chi}(0)\widehat{f}(0)]x^{1-s}\mathrm{d}^\times x\\ &+\int_1^\infty q^s\frac{\overline{\chi}(-1)\tau_{\overline{\chi}}}{q}[\theta(\widehat{\widehat{f}},\chi,x)-\chi(0)\widehat{\widehat{f}}(0)]x^s\mathrm{d}^\times x\end{aligned}$$

since $\widehat{\widehat{f}}(x)=f(-x)$ and f and χ have the same parity,

$$\theta(\widehat{\widehat{f}},\chi,x)=\chi(-1)\theta(f,\chi,x)$$

and using the identity $q=\chi(-1)\tau_\chi\tau_{\overline{\chi}}$ we obtain

$$\widetilde{\widehat{f}}(1-s)L(\overline{\chi},1-s)=\frac{q^s}{\chi(-1)\tau_\chi}\widetilde{f}(s)L(\chi,s)$$

$$q^{\frac{s}{2}}\widetilde{f}(s)L(\chi,s)=\varepsilon_\chi q^{\frac{1-s}{2}}\widetilde{\widehat{f}}(1-s)L(\overline{\chi},1-s) \tag{2.7}$$

where

$$\varepsilon_\chi=\chi(-1)\frac{\tau_\chi}{q^{1/2}}$$

Let us now take

$$f(x)=\begin{cases}e^{-\pi x^2}, & \text{si } \chi(-1)=1,\\ (e^{-\pi x^2})'=-2\pi xe^{-\pi x^2}, & \text{si } \chi(-1)=-1.\end{cases} \tag{2.8}$$

One has

$$\widehat{f}(x)=\begin{cases}f(x), & \text{if } \chi(-1)=1,\\ if(x), & \text{si } \chi(-1)=-1,\end{cases}$$

and therefore the Mellin transforms satisfy

$$\widetilde{f}(s)=\begin{cases}\frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}), & \text{si } \chi(-1)=1,\\ =-\frac{s-1}{2}\pi^{-\frac{s-1}{2}}\Gamma(\frac{s-1}{2})=-\pi\pi^{-\frac{s+1}{2}}\Gamma(\frac{s+1}{2}), & \text{si } \chi(-1)=-1.\end{cases}$$

This establish the functional equation.

3.2.2. *Gauss sums and Gamma functions.* In this section we prove the two facts about Gauss sums

THEOREM 2.3. *Let χ be a primitive character and*

$$\tau_\chi(m) = \sum_{a(q)} \chi(a) e(m\frac{a}{q})$$

be the associated Gauss sum. We have for any $m \in \mathbb{Z}$

$$\tau_\chi(m) = \overline{\chi}(m)\tau_\chi(1)$$

(in particular $\tau_\chi(m) = 0$ if $(m,q) \neq 1$) and for $(m,q) = 1$

$$|\tau_\chi(m)| = |\tau_\chi(1)| = q^{1/2}.$$

We have

$$\tau_\chi(1)\tau_{\overline{\chi}}(1) = \chi(-1)q.$$

PROOF. Suppose $(m,q) = 1$ the map $a \mapsto am \,(\mathrm{mod}\, q)$ is bijective with inverse $a \mapsto \overline{m}a$ (with $m.\overline{m} \equiv 1 \,(\mathrm{mod}\, q)$) and therefore by a change of variable

$$\tau_\chi(m) = \sum_{a(q)} \chi(a) e(m\frac{a}{q}) = \sum_{a(q)} \chi(\overline{m}a) e(\frac{a}{q}) = \chi(\overline{m})\tau_\chi(1).$$

Suppose now that $(m,q) = q' > 1$, the exponential $a \mapsto e(am/q)$ depends only on $a \,(\mathrm{mod}\, q/q')$ and we have

$$\tau_\chi(m) = \sum_{a'(q/q')} e(m\frac{a'}{q}) \sum_{b \equiv 1 \,(\mathrm{mod}\, q/q')} \chi(a'b) = 0$$

since $\chi_{|\ker(\mathrm{red}_{q/q'})}$ is non trivial and therefore

$$\sum_{b \equiv 1 \,(\mathrm{mod}\, q/q')} \chi(b) = 0.$$

Since $m \mapsto \tau_\chi(m)$ is the Fourier transform of χ we have by Plancherel formula

$$\sum_m |\tau_\chi(m)|^2 = \sum_{(m,q)=1} |\tau_\chi(m)|^2 = \varphi(q)|\tau_\chi(1)|^2 = q\sum_m |\chi(m)|^2 = q\varphi(q).$$

The last identity follows from the previous one and the fact that

$$\overline{\tau_\chi(1)} = \tau_{\overline{\chi}}(-1) = \chi(-1)\tau_{\overline{\chi}}(1).$$

$\square$

The Euler Γ function is an continuous analog of Gauss sums: let us recall $\Gamma(s)$ is defined for $\Re s > 1$ by

$$\Gamma(s) = \int_0^\infty e^{-x} x^s \mathrm{d}^\times x = \frac{1}{2}\int_0^\infty e^{-x^2} x^{s/2} \mathrm{d}^\times x$$

and extended to a meromorphic function on $\mathbb{C}$ via the function equation

$$\Gamma(s+1) = s\Gamma(s)$$

with simple poles at $-\mathbb{N}$. The Gamma function also satisfies the functional equation

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

and the duplication formula

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = \pi^{1/2}2^{1-2s}\Gamma(2s).$$

and in particular $\Gamma(s)$ has no zeros on $\mathbb{C}$. We also recall the Strilings formulae

$$\Gamma(s) = (\frac{2\pi}{s})^{1/2}(\frac{s}{e})^s(1+O_\delta(|s|^{-1})),\ |\arg s| < \pi - \delta, \delta > 0.$$

In particular for $t > 0$

$$\Gamma(\sigma+it) = t^{\sigma-1/2}\exp(-\pi t/2)(\frac{t}{e})^{it}(1+O_\sigma(|t|^{-1}))$$

and

$$\frac{\Gamma(\sigma+it)}{\Gamma(\beta-\sigma-it)} = O(|t|^{2\sigma-\beta}) \text{ as } t \to \infty.$$

regarding the logarythmic derivative we have

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\gamma - \sum_{n\geq 0}(\frac{1}{n+s} - \frac{1}{n+1}) = \log s - (2s)^{-1} + O(|s|^{-2}), |\arg s| < \pi - \delta, \delta > 0$$

4. Explicit formula

The relation between the Explicit formula (for convenience we repeat here the explicit formula for the riemann zeta function)

THEOREM (Explicit formula). *Let $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_{>0})$, and $\check{\varphi}(x) := x^{-1}\varphi(x^{-1})$. For χ, a non trivial primitive character one has*

$$\sum_n \Lambda(n)\chi(n)\varphi(n) + \sum_n \Lambda(n)\overline{\chi}(n)\check{\varphi}(n) = \varphi(1)\log q$$

$$-\sum_\rho \tilde{\varphi}(\rho) + \frac{1}{2\pi i}\int_{(1/2)}[\frac{L'_\infty(\chi,s)}{L_\infty(\chi,s)} + \frac{L'_\infty(\overline{\chi},1-s)}{L_\infty(\overline{\chi},1-s)}]\tilde{\varphi}(s)ds.$$

while for χ trivial and primitive (so that $L(\chi,s) = \zeta(s)$) one has

$$\sum_n \Lambda(n)\varphi(n) + \sum_n \Lambda(n)\check{\varphi}(n) = \int_0^\infty \varphi(x)dx + \int_0^\infty \varphi(x)\mathrm{d}^\times x$$

$$-\sum_\rho \tilde{\varphi}(\rho) + \frac{1}{2\pi i}\int_{(1/2)}[\frac{\zeta'_\infty(s)}{\zeta_\infty(s)} + \frac{\zeta'_\infty(1-s)}{\zeta_\infty(1-s)}]\tilde{\varphi}(s)ds.$$

Here the (absolutely converging) ρ-sum is along the non-trivial zeros of $L(\chi,s)$ counted with multiplicity.

PROOF. We proceed as of the Riemann zeta function explicit formula: we consider the integral

$$\frac{1}{2\pi i}\int_{(2)} -\frac{\Lambda'(\chi,s)}{\Lambda(\chi,s)}\tilde{\varphi}(s)ds$$
$$= -\frac{\varphi(1)}{2}\log q + \frac{1}{2\pi i}\int_{(2)}\tilde{\varphi}(s) + \sum_n \Lambda(n)\chi(n)\varphi(n) - \frac{1}{2\pi i}\int_{(2)}\frac{L'_\infty(\chi,s)}{L_\infty(\chi,s)}\tilde{\varphi}(s)ds.$$

We shift the last contour integral to $\Re s = 1/2$ hitting no poles on the way (since the only poles of $L_\infty(\chi,s)$ have non-positive real part).

Shifting the contour to $\Re e s = -1/2$, we hit simples poles at the zeros of $\Lambda(\chi,s)$ whose residues equal minus the multiplicity of the zero and we the usual convention we obtain

$$\frac{1}{2\pi i}\int_{(2)} -\frac{\Lambda'(\chi,s)}{\Lambda(\chi,s)}\tilde{\varphi}(s)ds = -\sum_\rho \tilde{\varphi}(\rho) + \frac{1}{2\pi i}\int_{(-1/2)} -\frac{\Lambda'(\chi,s)}{\Lambda(\chi,s)}\tilde{\varphi}(s)ds$$

and we apply the functional equation

$$\frac{\Lambda'(\chi,s)}{\Lambda(\chi,s)} = -\frac{\Lambda'(\overline{\chi},1-s)}{\Lambda(\overline{\chi},1-s)}$$

and make the change of variable $s \leftrightarrow 1-s$ bringing the contour back to $\Re s = 1-(-1/2) = 3/2$ and move further to $\Re s = 2$. $\square$

EXERCISE 2.4. *Prove rigorously the explicit formula above using Theorem 2.4*

5. Zeros of Dirichlet L-functions

Our aim will be to study the logarithmic derivative of $L(\chi,s)$ since for $\Re s > 1$ one has

$$-\frac{L'(\chi,s)}{L(\chi,s)} = \sum_{n\geq 1}\chi(n)\Lambda(n)n^{-s}.$$

In particular this function extend meromorphically to $\mathbb{C}$, its poles being located at the zeros of $L(\chi,s)$.

We observe first that

- $L(\chi,s) \neq 0$ for $\Re s > 1$ (since $L(\chi,s)$ is an Euler product in this half-plane),
- $\Lambda(\chi,s) \neq 0$ for $\Re s > 1$, since $L_\infty(\chi,s) \neq 0$ in this range;
- $\Lambda(\chi,s) \neq 0$ for $\Re s < 0$ by the functional equation,
- $L(\chi,s) \neq 0$ for $\Re s < 0$ excepted at the poles of $L_\infty(\chi,s) \neq 0$ (which are located at the even or odd negative integers depending on the parity of χ).

Definition 2.3. *The non-trivial zeros of $L(\chi, s)$ are by definition the zeros of $L(\chi, s)$ contained in $\{\Re s \in [0,1]\}$ or equivalently the zeros of $\Lambda(\chi, s)$. We denote their multiset (the set of zeros each zero being assigned its multiplicity) by*

$$\mathcal{Z}(\chi) = \{\rho = \sigma + i\gamma,\ L(\chi, \rho) = 0,\ \sigma \in [0,1]\}.$$

Similarly for any $T \geq 0$ we set

$$\mathcal{Z}(\chi, T) = \{\rho = \sigma + i\gamma,\ L(\chi, \rho) = 0,\ \sigma \in [0,1],\ |t| \leq T\}$$

and denote by

$$N(\chi, T) = |\mathcal{Z}(\chi, T)|$$

the size of this multiset.

Observe that by conjugation the functional equation the maps $\rho \mapsto \overline{\rho}$ and $\rho \mapsto 1 - \rho$ are bijective from $\mathcal{Z}(\chi, T)$ to $\mathcal{Z}(\overline{\chi}, T)$. It follows that the map $\rho \mapsto \rho^* = 1 - \overline{\rho}$ is bijection on $\mathcal{Z}(\chi, T)$ to itself.

We will prove the following

Theorem 2.4. *For any $T \geq 0$, one has*

$$N(\chi, T) - N(\chi, T-1) \ll \log(q(2+T)), \tag{2.9}$$

and

$$N(\chi, T) \ll (2+T)\log(q(2+T)) \tag{2.10}$$

Secondly we will give a non-trivial region for the location of the zeros

Theorem 2.5 (Landau). *There exist an some absolute $C > 0$ such that for any primitive character χ, the set of non-trivial zeros of $L(\chi, s)$ is contained in the domain*

$$\{s = \sigma + it,\ \sigma \leq 1 - \frac{C}{\log(q(2+|t|))}\} \tag{2.11}$$

excepted possibly if χ is quadratic (non-trivial). In that later case, there is at most one such zero contained inside this domain; this zero if it exists, is simple, real, stricly less than 1 and is called the exceptional zero. Moreover there is at most one χ of modulus q having such an exceptional zero. If this zero ever exist, it will be noted ρ_q^{exp} and the associated character χ^{exp}.

Conjecture (GRH). *One has*

$$\mathcal{Z}(\chi) \subset \{\Re s = 1/2\}$$

5.1. Counting zeros.

5.1.1. *Functions of order at most* 1.

DEFINITION 2.4. *An holomorphic function on $\mathbb{C}$, $L(s)$ is of order ≤ 1 if for any $\alpha > 1$, one has*

$$L(s) \ll_\alpha \exp(|s|^\alpha).$$

THEOREM 2.6. *Let $L(s)$ be of order at most* 1*, then for any $\alpha > 1$*

$$N(L,R) = |\{\rho,\ |\rho| \leq R,\ L(\rho) = 0\}| \ll_{\alpha,L} R^\alpha.$$

In particular

$$\sum_{L(\rho)=0} \frac{1}{|\rho|^\alpha + 1} < \infty.$$

THEOREM 2.7 (Hadamard factorization). *Let $L(s)$ be of order at most* 1 *and let m_0 be the order of vanishing of $L(s)$ at $s = 0$, one has*

$$L(s) = \exp(a + bs)s^{m_0} \prod_{L(\rho)=0,\ \rho\neq 0} (1 - \frac{s}{\rho}) \exp(s/\rho)$$

(each factor occuring with multiplicity equal to the order of the zero ρ). The convergence being uniform on compact set avoiding the set of zeros of $L(s)$. Taking logarithmic derivative, one also has

$$\frac{L'(s)}{L(s)} = b + \frac{m_0}{s} + \sum_{\rho\neq 0} \frac{1}{s-\rho} + \frac{1}{\rho}$$

(each factor occuring with multiplicity equal to the order of the zero ρ). The convergence is normal on on compact set avoiding the set of zeros of $L(s)$.

REMARK 2.5. Observe that for s in a compact set avoiding the zeros of $L(s)$

$$\frac{1}{s-\rho} + \frac{1}{\rho} = \frac{s}{(s-\rho)\rho} = O(1/|\rho|^2)$$

hence the series is converging.

5.2. Application to L-functions.

PROPOSITION 2.1. *The function $(s(1-s))^{\chi=1}\Lambda(\chi,s)$ is of order ≤ 1.*

PROOF. We may assume that $\Re s \geq 1/2$. On has

$$|\Lambda(\chi,s)| \leq \int_1^\infty (|\theta(f,\chi,x)| + |\theta(\hat{f},\overline{\chi},x)|(x^\sigma + x^{1-\sigma})\mathrm{d}^\times x$$
$$\ll \int_1^\infty \exp(-\pi x^2)x^{\sigma+1}\mathrm{d}^\times x \ll \pi^{\sigma/2}\Gamma(\frac{\sigma+1}{2}) \ll_\alpha \exp(\sigma^\alpha).$$

for any $\alpha > 1$ by Stirlings formula. $\square$

Let m_χ be the residue of $\frac{\Lambda'(\chi,s)}{\Lambda'(\chi,s)}$ at $s = 1$: m_χ is either -1 if $\chi = 1$ or the order of vanishing of $L(\chi,s)$ at $s = 1$ if $\chi \neq 1$. Since

$$\overline{\Lambda(\chi,s)} = \Lambda(\overline{\chi},\overline{s}),$$

we have $m_\chi = m_{\overline{\chi}}$ and because of the functional equations

$$\Lambda(\chi, s) = \varepsilon_\chi \Lambda(\overline{\chi}, 1-s),$$

m_χ is also the residue of $\frac{\Lambda'(\chi,s)}{\Lambda'(\chi,s)}$ at $s = 0$.

By Hadamard's factorization theorem, we have

$$\begin{aligned}
\frac{\Lambda'(\chi, s)}{\Lambda(\chi, s)} &= \frac{L'(\chi, s)}{L(\chi, s)} + \frac{L_\infty(\chi, s)}{L_\infty(\chi, s)} + \frac{1}{2}\log q \\
&= b_\chi + \frac{m_\chi}{s} + \frac{m_\chi}{s-1} + \sum_{\rho \neq 0,1} \frac{1}{s-\rho} + \frac{1}{\rho}
\end{aligned} \tag{2.12}$$

for some constant b_χ depending on χ. As $s \to 0$ we have

$$\frac{\Lambda'(\chi, s)}{\Lambda(\chi, s)} = \frac{m_\chi}{s} + b_x - m_\chi + O(s)$$

therefore (taking the complex conjugate)

$$\overline{b_\chi} = b_{\overline{\chi}} \text{ that is } 2\Re b_\chi = b_\chi + b_{\overline{\chi}}.$$

We will use these relations to control the zeros of $L(\chi, s)$ as the logarithmic derivative $L'(\chi, s)/L(\chi, s))$. However we will be also interested in the dependency of this control with q. For this we will need to eliminate the constant b_χ

By the functional equation, the sum of the logarithmic derivatives of $\Lambda(\chi, s)$ and of $\Lambda(\overline{\chi}, 1-s)$ is zero; it follows that for s not a pole

$$b_\chi + b_{\overline{\chi}} = 2\Re b_\chi = -\sum_{\rho \neq 0,1} \frac{1}{s-\rho} + \frac{1}{1-s-\overline{\rho}} + \frac{1}{\rho} + \frac{1}{\overline{\rho}}.$$

We have

$$\Re\frac{1}{\rho} = \frac{\beta}{|\rho|^2} \leq \frac{1}{|\rho|^2}$$

and therefore the series

$$\sum_{\rho \neq 0} \frac{1}{\rho} + \frac{1}{\overline{\rho}} \text{ and } \sum_{\rho \neq 0,1} \frac{1}{s-\rho} + \frac{1}{1-s-\overline{\rho}}$$

are both converging. Since $\mathcal{Z}(\chi)$ is invariant under $\rho \mapsto 1 - \overline{\rho}$ one has

$$\sum_{\rho \neq 0,1} \frac{1}{s-\rho} - \frac{1}{s-(1-\overline{\rho})} = 0 \text{ and } \Re b_\chi = -\sum_{\rho \neq 0,1} \Re(\frac{1}{\rho}).$$

It follows that

Lemma 2.1. *For any $s \notin \mathcal{Z}(\chi)$*

$$-\Re\frac{\Lambda'(\chi, s)}{\Lambda(\chi, s)} = \frac{1}{2}\log q - \Re(\frac{m_\chi}{s} + \frac{m_\chi}{s-1}) - \sum_{\rho \neq 0,1} \Re\frac{1}{s-\rho}. \tag{2.13}$$

In the sequel, we will use several times the following positivity statement: let $s=\sigma+it$ satisfying $\sigma>1$; writing any zero $\rho=\beta+i\gamma$ with $\beta\leq 1$ one has

$$\Re(\frac{1}{s-\rho})=\frac{\sigma-\beta}{|\sigma-\beta|^2+|t-\gamma|^2}\geq 0.$$

5.2.1. *Proof of Thm. 2.4.* Let us choose $s=3+iT$, then the lefthand side of (2.13) is (by Stirling's formula)

$$-\Re\frac{L'_\infty(\chi,s)}{L_\infty(\chi,s)}-\Re\frac{L'(\chi,s)}{L(\chi,s)}=O(\log(2+|T|))+O(1);$$

we have therefore (since $2\leq 3-\beta\leq 3$)

$$\sum_\rho\Re\frac{1}{s-\rho}=\sum_\rho\frac{3-\beta}{(3-\beta)^2+(T-\gamma)^2}\ll\log(q(2+|T|)).$$

Since all the terms on the left-hand side are non-negative we deduce that

$$|\{\beta+i\gamma,\ |\gamma-T|\leq 1\}|\ll\sum_\rho\frac{3-\beta}{(3-\beta)^2+(T-\gamma)^2}\ll\log(q(2+|T|)).$$

$\square$

Controling the number of zeros enable us to control to control the size of $\frac{L'(\chi,s)}{L(\chi,s)}$; the following Corollary is an immediate consequence of (2.12) and (2.9):

COROLLARY 2.1. *For any $s\notin\mathcal{Z}(\chi)\cup\{0,1\}$ and such that $|\Re s|\leq 10$ one has*

$$\frac{\Lambda'(\chi,s)}{\Lambda(\chi,s)}=m_\chi(\frac{1}{s}+\frac{1}{s-1})+\sum_{\substack{\rho\neq 0,1\\|s-\rho|\leq 1}}\frac{1}{s-\rho}+\frac{1}{\rho}+O\chi(\log(2+|s|)).$$

In particular there exists an sequence $(t_n)_{n\geq 1}$ with $t_n\rightarrow+\infty$ such that for $|\sigma|\leq 10$ and $n\geq 1$, one has

$$\frac{L'(\chi,\sigma\pm t_n)}{L(\chi,\sigma\pm t_n)},\ \frac{\Lambda'(\chi,\sigma\pm t_n)}{\Lambda(\chi,\sigma\pm t_n)}\ll_\chi\log(2+t_n).$$

The above estimates do not display explicitly the dependency on q which is important for us; the next corollary takes care of this upon considering the real part:

COROLLARY 2.1. *For $s=\sigma+it\notin\mathcal{Z}(\chi)$ with $|\sigma|\leq 10$, one has*

$$\begin{aligned}-\Re\frac{L'(\chi,s)}{L(\chi,s)}&=-m_\chi(\frac{1}{s-1}+\frac{1}{s})-\sum_{\rho\neq 0,1}\Re\frac{1}{s-\rho}+O(\log(q(2+|t|)))\\&=-m_\chi(\frac{1}{s-1}+\frac{1}{s})-\sum_{\rho\neq 0,1,\ |\rho-s|\leq 1}\Re\frac{1}{s-\rho}+O(\log(q(2+|t|)))\end{aligned}$$

5.3. Proof of Thm. 2.5. Let $\rho=\beta+i\gamma$ be a zero of $L(\chi,s)$ with $\beta\geq 1/2$.

The basic idea which goes back to HdVP is to construct a Dirichlet series

$$L(\chi,\gamma,s)=1+\sum_{n>1}\frac{a(\chi,\gamma,n)}{n^s}$$

with non-negative coefficients and which contains $L(\chi,s+i\gamma)$ as a factor with high multiplicity and therefore a high order real zero at $s=\beta$. If β is too close to 1 this will contradict the fact that $L(\chi,\gamma,s)$ has to be greater than 1 near $s=1$.

Here we will proceed with the logarithmic derivative instead (as Hadamard and de la Vallée-Poussin did originaly).

Let

$$L(\chi,\gamma,s):=\zeta(s)[L(\chi,s+i\gamma)L(\overline{\chi},s-i\gamma)\zeta(s)]^2L(\chi^2,s+2i\gamma)L(\overline{\chi}^2,s-2i\gamma)$$

We have for $\Re s>1$,

$$-\frac{L'(\chi,\gamma,s)}{L(\chi,\gamma,s)}=\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}(1+2(\frac{\chi(n)}{n^{i\gamma}}+\frac{\overline{\chi}(n)}{n^{-i\gamma}}+1)+\frac{\chi^2(n)}{n^{i2\gamma}}+\frac{\overline{\chi}^2(n)}{n^{-i2\gamma}})$$
$$=\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}(1+\frac{\chi(n)}{n^{i\gamma}}+\frac{\overline{\chi}(n)}{n^{-i\gamma}})^2$$

has non-negative real coefficients.

- Let us assume first that $\chi^2\neq 1$, then the only possible pole of $L(\chi,\gamma,s)$ in the half-plane $\{\Re s\geq 1/2\}$ is at $s=1$ and has order ≤ 3; therefore for $s>1$ we have

$$0\leq -\frac{L'(\chi,\gamma,s)}{L(\chi,\gamma,s)}=\frac{3}{s-1}-2(\sum_{\rho'}\Re\frac{1}{s+i\gamma-\rho'}+\sum_{\overline{\rho}'}\Re\frac{1}{s-i\gamma-\overline{\rho}'})$$
$$-\sum_{\rho''}\Re\frac{1}{s+2i\gamma-\rho''}-\sum_{\overline{\rho}''}\Re\frac{1}{s-2i\gamma-\overline{\rho}''}+O(\log(q(2+|\gamma|))),$$

and since we may remove any negative term from the right-hand side, we deduce that

$$0\leq\frac{3}{s-1}-2\Re(\frac{1}{s+i\gamma-\rho}+\frac{1}{s-i\gamma-\overline{\rho}})+O(\log(q(2+|\gamma|))).$$

We have therefore

$$\frac{4}{s-\beta}\leq\frac{3}{s-1}+O(\log(q(2+|\gamma|)));$$

since $4>3$ implies the bound (2.11) by taking $s=1+c/\log(q(2+|\gamma|))$ for $c>0$ sufficiently small.

- We assume now that $\chi^2=1$ but χ is non trivial; we have

$$L(\chi^2,s)=\zeta(s)\prod_{p|q}(1-\frac{1}{p^s})$$

so that in the half-plane $\{\Re s \geq 1/2\}$, the function $L(\chi, \gamma, s)$ has a triple pole at $s = 1$ and two simple poles at $s = 1 \pm 2i\gamma$. Therefore by the previous reasonning we have

$$\frac{4}{s-\beta} \leq \frac{3}{s-1} + \frac{1}{s-1-2i\gamma} + \frac{1}{s-1-2i\gamma} + O(\log(q(2+|\gamma|)))$$

$$\frac{4}{s-\beta} \leq \frac{3}{s-1} + \frac{1}{s-1-2i\gamma} + \frac{1}{s-1-2i\gamma} + O(\log(q(2+|\gamma|)))$$

$$\frac{4}{s-\beta} \leq \frac{3}{s-1} + \frac{2(s-1)}{(s-1)^2+4\gamma^2} + O(\log(q(2+|\gamma|))).$$

Taking s of the form $s = 1+c/\log(q(2+|\gamma|))$, we see that for if $|\gamma| \geq \frac{1}{\log q}$ the middle term on the right-hand side of the above inequality is $O(\log(q(2+|\gamma|)))$ and is absorbed by the third term and we conclude that $L(\chi, s)$ has not zero in the range (2.11) and satisfying $|\gamma| \geq \frac{1}{\log q}$.

To deal with the range $|\gamma| \leq 1/\log q$ we consider another auxilliary L-function: we remark that since χ take values in $0, \pm 1$, we have for any n

$$1 + \chi(n) \geq 0$$

and therefore considering the L-function $\zeta(s)L(\chi, s)$ and the negative of its logarithmic derivative have non-negative coefficient; we deduce as before

$$\sum_{\rho} \Re\frac{1}{s-\rho} \leq \frac{1}{s-1} + O(\log q)$$

Consider now $\rho = \beta + i\gamma$ with $\gamma \neq 0$ then $\overline{\rho} = \beta - i\gamma$ is also a zero of $L(\chi, s)$ and we therefore have

$$2\frac{s-\beta}{(s-\beta)^2+\gamma^2} \leq \frac{1}{s-1} + O(\log q).$$

we deduce from this a contradiction (since $2 > 1$) if

$$1 - \beta \ll 1/\log q, \ |\gamma| \leq 1/\log q.$$

Suppose that $\gamma = 0$ and let m_β be the multiplicity of this zero: we have

$$\frac{m_\beta}{s-\beta} \leq \frac{1}{s-1} + O(\log q)$$

which yield a contradiction if $m_\beta \geq 2$ and $1 - \beta \ll 1/\log q$. therefore the exceptional zero must be simple.

If we assume that $\chi = 1$, then $L(\chi, \gamma, s)$ has also poles at $s = 1 \pm i\gamma$ and $s = 1 \pm 2i\gamma$ and as soon as $\gamma \gg 1$ their contribution is bounded by $O(1)$ and is absorbed in the $\log(2 + |\gamma|)$ term.

□

The next result shows that the exceptional zero is really exceptional:

THEOREM 2.8 (Landau-Page). *Let $Q \geq 1$, then*

$$\prod_{q \leq Q} \prod_{\chi(q)} L(\chi, s)$$

has at most one real simple zero in the range

$$[1 - C/\log Q, 1].$$

PROOF. suppose that $\chi_0 \,(\mathrm{mod}\, q_0)$ has a real zero $\beta_0 \in [1 - C/\log q, 1]$, then we consider for $\chi \neq \chi_0$ quadratic

$$L(s) = \zeta(s)L(\chi, s)L(\chi_0, s)L(\chi\chi_0, s)$$

which as non-negative coefficients. We have

$$\frac{1}{s-\beta_0} + \frac{1}{s-\beta} \leq \frac{1}{s-1} + C_1 \log q$$

□

6. Siegel's theorem

Observe that Landau's theorem does not a apriori eliminate the possibility that for χ quadratic non-trivial $L(\chi, s)$ has a zero at $s = 1$! In the course "introduction to Analytic number Theory" we gave a proof that $L(\chi, 1) \neq 0$ (which is a crucial step toward the proof of Dirichlet prime number theorem in arithmetic progressions).

As a preparation for the proof of Siegel's theorem we give another proof of this fact

THEOREM 2.9 (Dirichlet). *For any non-trivial quadratic character χ, one has*

$$L(\chi, 1) \neq 0.$$

PROOF. We consider

$$\zeta(s)L(\chi, s) = \sum_{n \geq 1} \frac{1 * \chi(n)}{n^s}.$$

we remark that $1 * \chi(n) \geq 0$ and even that for any n, $1 * \chi(n^2) \geq 1$. In particular if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}_{>0})$ is non-negative, we have

$$\sum_n 1 * \chi(n)\varphi(\frac{n}{x}) \gg_\varphi x^{1/2}.$$

On the other hand, by inverse Mellin transform we have

$$\sum_n 1 * \chi(n)\varphi(\frac{n}{x}) = \frac{1}{2\pi i}\int_{(2)} \zeta(s)L(\chi, s)\tilde{\varphi}(s)x^s ds.$$

Let us shift the previous contour to $\Re s = 0$ (this being justified by the lemma below), the function $\zeta(s)L(\chi, s)\tilde{\varphi}(s)x^s$ is holomorphic on the strip $\{s,\ \Re s \in [0, 2]\}$ excepted possibly for an at most simple pole at $s = 1$ (coming from the pole of $\zeta(s)$) with resiue equal to

$$L(\chi, 1)\tilde{\varphi}(1)x.$$

We have therefore

$$x^{1/2} \ll \sum_n 1 * \chi(n)\varphi(\frac{n}{x}) = L(\chi,1)\tilde{\varphi}(1)x + \frac{1}{2\pi i}\int_{(1/4)} \zeta(s)L(\chi,s)\tilde{\varphi}(s)x^s ds.$$

By the following lemma the last integral is bounded by $O(1)$ giving

$$x^{1/2} \ll L(\chi,1)\tilde{\varphi}(1)x + O(1)$$

which is a contradiction if $L(\chi,1)=0$. □

Lemma 2.2. *There exists $A \geq 0$ such that*

$$\zeta(\sigma+it),\ L(\chi,\sigma+it) \ll |t|^A$$

uniformly for $\sigma \in [0,+\infty]$ and $|t| \geq 1$.

Proof. We deal with $L(\chi,s)$. Let us recall that

$$M(\chi,x) = \sum_{n\leq x}\chi(n) = O(q)$$

therefore for $\Re s > 0$

$$\sum_n \frac{\chi(n)}{n^s} = [M(\chi,x)x^{-s}]_1^\infty - s\int_1^\infty M(\chi,x)x^{-s-1}dx = 1 + O(q\frac{|s|}{\sigma}).$$

The same reasoning with $M(\chi,x)$ replaced by

$$M(1,x) = \sum_{n\leq x} 1 = [x] = x - \{x\}$$

yield the result for $\zeta(s)$. This lemma control the growth of $L(\chi,s)$ in vertical bands for $\Re s \geq 1/2$. By the functional equation

$$L(\chi,s) = \varepsilon_\chi \frac{L_\infty(\overline{\chi},1-s)}{L_\infty(\chi,s)}L(\overline{\chi},1-s)$$

along Stirling's formula enable us to control the growth for $\Re s \leq 1/2$. □

We will also need the following more precise version of this bound when s is close to 1:

Lemma 2.3. *Let χ be non-trivial, we have*

$$L(\chi,s), \frac{L'(\chi,s)}{\log q} \ll \log q + |s|$$

uniformly for $|\Re s - 1| \ll 1/\log q$.

Proof. We have

$$L(\chi,s) = \sum_{n\leq X}\frac{\chi(n)}{n^s} + [M(\chi,x)x^{-s}]_X^\infty - s\int_X^\infty M(\chi,x)x^{-s-1}dx$$

$$= O(\log X + \delta_{\sigma\neq 1}\frac{X^{1-\sigma}-1}{1-\sigma}) + O(|s|qX^{-\sigma}) = O(\log q + |s|)$$

on choosing $X = q$. The proof for the derivative is the same. □

EXERCISE 2.5. *Prove that for* $\Re s = 0$,

$$L(\chi, s) = O((1 + |s|)^{1/2} q^{1/2} (\log q + |s|))$$

and prove that

$$L(\chi, 1) \gg \frac{1}{q^{1/2} \log q} \tag{2.14}$$

and that the largest real zero β_χ *of* $L(\chi, 1)$ *satisfies*

$$1 - \beta_\chi \gg \frac{1}{q^{1/2} \log^3 q}.$$

6.1. Proof of Siegel's theorem. In this section we establish Siegel's theorem which locates the exceptional zero more precisely:

THEOREM 2.10 (Siegel). *For any* $\varepsilon > 0$, *there is a constant* q_ε *such that for any* $q \geq q_\varepsilon$, *and any primitive quadratic character,* $L(\chi, s)$ *has no zero in the interval*

$$[1 - q^{-\varepsilon}, 1].$$

Moreover one has

$$L(\chi, 1) \gg q^{-\varepsilon}$$

PROOF. Let χ, χ' be two distinct and non-trivial primitive quadratic characters and let

$$L(\chi, \chi', s) = \zeta(s) L(\chi, s) L(\chi', s) L(\chi\chi', s);$$

For $\Re s > 1$, this is the Dirichlet series associated to the convolution

$$1 * \chi * \chi' * \chi\chi'.$$

LEMMA 2.4. *For any* $n \geq 1$,

$$a(n) := 1 * \chi * \chi' * \chi\chi'(n) \geq 0.$$

PROOF. Observe that for $n = p$ a prime

$$1 * \chi * \chi' * \chi\chi'(p) = 1 + \chi(p) + \chi'(p) + \chi\chi'(p) = (1 + \chi(p))(1 + \chi'(p)) \geq 0.$$

To prove the general case we observe that

$$\log L(\chi, \chi', s) = \sum_p \log \zeta_p(s) + \log L(\chi, s) + \log L(\chi', s) + \log L(\chi', s)$$

$$\sum_p \sum_{\alpha \geq 1} \frac{1}{\alpha p^{\alpha s}} (1 + \chi(p^\alpha) + \chi'(p^\alpha) + \chi(p^\alpha) + \chi'(p^\alpha))$$

and

$$(1 + \chi(p^\alpha) + \chi'(p^\alpha) + \chi(p^\alpha) + \chi'(p^\alpha)) = (1 + \chi(p^\alpha))(1 + \chi'(p^\alpha)) \geq 0$$

and we conclude since as a formal series we have

$$L(\chi, \chi', s) = \exp(\log L(\chi, \chi', s)).$$

Let φ be a non-negative smooth function taking value 1 on $[0, 1]$ and 0 on $[1, \infty[$. □

Given $\beta' \in [3/4, 1[$ and $x \geq 1$ consider the sum

$$S(\beta', \varphi) = \sum_{n \geq 1} \frac{a(n)}{n^{\beta'}} \varphi(n/x) = \frac{1}{2\pi i} \int_{(2)} L(\chi, \chi', s + \beta') \tilde{\varphi}(s) x^s ds.$$

Since $a(1) = 1$ by the previous lemma we have

$$S(\beta', \varphi) \geq a(1)\varphi(1/x) \gg 1.$$

On the other hand, shifting the contour to $\Re s = 1/2 - \beta' \in [-1/2, -1/4[$ we pick a pole at $s = 1 - \beta'$ (indeed $L(\chi, \chi', s)$ has a simple pole at $s = 1$ since $L(\chi, 1)L(\chi', 1)L(\chi\chi', 1) \neq 0$) and at $s = 0$ (from $\tilde{\varphi}(s)$ which has residue $\tilde{\varphi}(1) = \int_0^\infty \varphi(x)dx$) therefore

$$1 \leq L(\chi, 1)L(\chi', 1)L(\chi\chi', 1)\tilde{\varphi}(1 - \beta')x^{1-\beta'} + L(\chi, \chi', \beta')\tilde{\varphi}(1) + \frac{1}{2\pi i} \int_{(1/2-\beta')} L(\chi, \chi', s + \beta')\tilde{\varphi}(s)x^s ds$$

and therefore

$$1 \leq L(\chi, 1)L(\chi', 1)L(\chi\chi', 1)\tilde{\varphi}(1-\beta')x^{1-\beta'} + L(\chi, \chi', \beta')\tilde{\varphi}(1) + O((qq')^2 x^{-1/4}).$$

Take $x = (qq')^9$ and therefore for qq' sufficiently large $O((qq')^2 x^{-1/4}) \leq 1/2$ and therefore

$$1/2 \leq L(\chi, 1)L(\chi', 1)L(\chi\chi', 1)\tilde{\varphi}(1 - \beta')x^{1-\beta'} + L(\chi, \chi', \beta')\tilde{\varphi}(1)$$

and if $L(\chi', \beta') = 0$ we obtain

$$1/2 \leq L(\chi, 1)L(\chi', 1)L(\chi\chi', 1)\tilde{\varphi}(1 - \beta')x^{1-\beta'}.$$

Now

$$\tilde{\varphi}(1 - \beta') = \int_0^\infty \varphi(t)t^{1-\beta'} dt/t = -\frac{1}{1 - \beta'}\tilde{\varphi}'(2 - \beta') \ll \frac{1}{1 - \beta'}$$

and by (2.3) we obtain that

$$L(\chi, 1) \gg (1 - \beta')\frac{(qq')^{9(1-\beta')}}{\log^2(qq')}.$$

For any $0 < \varepsilon < 1/4$, suppose there exist $\chi' = \chi'_\varepsilon$ so that $L(\chi', s)$ has at least one zero β'_ε in the interval $[1 - \varepsilon/10, 1[$ (if there are none we are finished !) we have therefore

$$L(\chi, 1) \gg_\varepsilon q^{-\varepsilon}.$$

Let $\beta_\chi < 1$ be the real zero of $L(\chi, s)$ closest to 1. We assume that

$$1 - 1/\log q \leq \beta_\chi,$$

then

$$\frac{L(\chi, 1)}{1 - \beta_\chi} = L'(\chi, \sigma) \ll \log^2 q$$

for $\sigma \in [1 - 1/\log q, 1]$ and therefore the exists $\beta \in [\beta_\chi, 1]$

$$1 - \beta_\chi = \frac{L(\chi, 1)}{L'(\chi, \beta)} \gg \frac{L(\chi, 1)}{\log^2 q} \gg q^{-2\varepsilon}.$$

Remark 2.6. A curious feature of Siegel's theorem is that it is non effective: the theorem states that for any $\varepsilon > 0$ there exists $q(\varepsilon)$ such that $L(\chi, s)$ the largest real zero of $L(\chi, s)$ is bounded by $1 - q^{-\varepsilon}$; however due to the shape of the proof, $q(\varepsilon)$ depends on the existence of some χ' having a real zero $\geq 1 - \varepsilon$, the function $q(\varepsilon)$ cannot be computed effectively in terms of ε if ε is too small : as soon as $\varepsilon < 1/2$, when $\varepsilon = 1/2$ the lower bound (2.14) is effective. Similarly we know that

$$L(\chi, 1) \geq c(\varepsilon) q^{-\varepsilon}$$

for some $c(\varepsilon) > 0$ but the value of $c(\varepsilon)$ is not explicit if $\varepsilon < 1/2$.

7. Proof of the Siegel-Walfisz theorem

in this section, we prove Teorem 2 using our previous results on zeros of L-functions.

Let $x^{-1/2} < \Delta < 1/2$ to be determined later and let φ be a smooth function non-negative which is bounded by 1, compactly supported on zero on

$$[x(1 - 2\Delta), x(1 + \Delta)]$$

increasing on on $[x(1 - 2\Delta), x(1 - \Delta)]$, constant equal to 1 on $[x(1 - \Delta), x]$ decreasing on $[x, x(1 + \Delta)]$ and zero from there and which satisfies for any $k \geq 1$

$$t^k \varphi^{(k)}(t) \ll \Delta^{-k}. \tag{2.15}$$

We have

$$\psi(x; q, a) = \sum_{\substack{n \equiv a \,(\mathrm{mod}\, q) \\ n \leq x}} \Lambda(n) = \sum_{n \equiv a \,(\mathrm{mod}\, q)} \Lambda(n)\varphi(n) + O(\Delta \frac{x \log x}{\varphi(q)} + 1)$$

and we have

$$\begin{aligned} \sum_{\substack{n \equiv a \,(\mathrm{mod}\, q) \\ n \leq x}} \Lambda(n)\varphi(n) &= \frac{1}{\varphi(q)} \sum_{(n,q)=1} \Lambda(n)\varphi(n) + \max_{\chi \,(\mathrm{mod}\, q) \neq 1} |\sum_n \Lambda(n)\chi(n)\varphi(n)| \\ &= \frac{1}{\varphi(q)} \sum_{\substack{n \leq x \\ (n,q)=1}} \Lambda(n) + O(\Delta \frac{x \log x}{\varphi(q)}) \\ &\quad + \max_{\chi \,(\mathrm{mod}\, q) \neq 1} |\sum_n \Lambda(n)\chi(n)\varphi(n)| \end{aligned}$$

Let $\tilde{\chi}\,(\mathrm{mod}\,q')$ be the primitive character associated with $\chi\,(\mathrm{mod}\,q)$, one has

$$\sum_n \Lambda(n)\chi(n)\varphi(n) - \sum_n \Lambda(n)\tilde{\chi}(n)\varphi(n) = O(\sum_{p^\alpha|q}\log p) = O(\log q)$$

and so we may assume that χ is primitive of modulus $1 < q'|q$.

By the explicit formula we have (simplify notation we assume $q' = q$)

$$\sum_n \Lambda(n)\chi(n)\varphi(n) =$$

$$\delta_{\chi=1}[\tilde{\varphi}(1)+\tilde{\varphi}(0)] + \varphi(1)\log q$$

$$-\sum_\rho \tilde{\varphi}(\rho) + \frac{1}{2\pi i}\int\limits_{(1/2)}[\frac{L'_\infty(\chi,s)}{L_\infty(\chi,s)} + \frac{L'_\infty(\overline{\chi},1-s)}{L_\infty(\overline{\chi},1-s)}]\tilde{\varphi}(s)ds.$$

Using the relation for $k \geq 1$

$$\tilde{\varphi}(s) = \frac{(-1)^k}{s\cdots(s+k-1)}\widetilde{\varphi^{(k)}}(s+k)$$

(2.15) and the fact that $\varphi^{(k)}$ is supported on $[x(1-2\Delta), x(1-\Delta)] \cup [x, x(1+\Delta)]$ we deduce that

$$\tilde{\varphi}(s) \ll \min_{k\leq 2}(|s|\Delta)^{-k}(\Delta x^\sigma + x^{\sigma/2})$$

unformly for $\sigma = \Re s \in [1/2, 2]$. We have

$$\tilde{\varphi}(1)+\tilde{\varphi}(0) = \int_0^\infty \varphi(t)(1+1/t)dt = x + O(\Delta x)$$

and

$$\frac{1}{2\pi i}\int\limits_{(1/2)}[\frac{L'_\infty(\chi,s)}{L_\infty(\chi,s)} + \frac{L'_\infty(\overline{\chi},1-s)}{L_\infty(\overline{\chi},1-s)}]\tilde{\varphi}(s)ds = O(\Delta^{-2}x^{1/2}).$$

It remains to estimate

$$\sum_\rho \tilde{\varphi}(\rho) \ll \sum_\rho x^\beta \min_{k\leq 2}(|\rho|\Delta)^{-k}.$$

By the zero-free region (Theorem 2.5) and the zero counting bound (2.9) this is bounded by

$$\ll \delta_{\chi=\chi^{exp}} x^{\beta^{exp}} + x\sum_\rho x^{-\frac{c}{\log(q|\rho|)}} \min_{k\leq 2}(|\rho|\Delta)^{-k}$$

$$\ll \delta_{\chi=\chi^{exp}} x^{\beta^{exp}} + x\Delta^{-2}\log q \exp(-c'\sqrt{\log x})$$

for some $c' > 0$. Here β^{exp} denote the exceptional zero if it exists.

Choosing $\Delta = \exp(-c''\sqrt{\log x})$ for $c'' > 0$ sufficiently small we obtain that (for $q \leq x$)

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O(x\exp(-c\sqrt{\log x})) + O(\frac{x}{\varphi(q)}x^{\beta^{exp}-1}).$$

By Siegel's theorem we have for any $\varepsilon > 0$

$$x^{\beta^{exp}-1} \leq \exp(-c(\varepsilon)\frac{\log x}{q^{\varepsilon}}) = O_{\varepsilon}(\exp(-c\sqrt{\log x}))$$

if $q \leq (\log x)^A$ for any $A \geq 1$ (upon choosing $\varepsilon = \varepsilon(A)$ small enough).

To resume we have proven that, if for some $A \geq 1$ we have $q \leq (\log x)^A$ then we have

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O_A(x \exp(-c_A\sqrt{\log x}))$$

for some $c(A) > 0$ depending on A

This statement implies the original Siegel-Walfisz.

Remark 2.7. As we have seen, the eventual presence of an exceptional zero significantly weaken the uniformity of q wrt x. Indeed is the exceptional zero did not exist one would have

$$\psi(x; q, a) = \frac{x}{\varphi(q)} + O(x \exp(-c\sqrt{\log x}))$$

for some absolute constant $c > 0$ and therefore $\psi(x; q, a) \simeq \frac{x}{\varphi(q)}$ uniformly for

$$q \leq \exp(C\sqrt{\log x})$$

instead of $q \leq (\log x)^A$.

However the main problem with the Siegel zero is the ineffectivity of the constant $c(\varepsilon)$ when $\varepsilon < 1/2$.

A major problem is to prove that the exceptional zero does not exists at all.