TOPICS IN NUMBER THEORY - EXERCISE SHEET I

École Polytechnique Fédérale de Lausanne

Exercise 1 **(Non-vanishing of Dirichlet L-functions on the line $\Re(s) = 1$)**
Let $q \geq 1$ and let χ be a Dirichlet character modulo q. Let also $L(\chi, s)$ be the Dirichlet L-function defined for $\Re(s) > 1$ by

$$L(\chi, s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}.$$

The goal of this exercise is to prove that $L(\chi, s) \neq 0$ for $\Re(s) = 1$. This generalizes the result of Dirichlet which asserts that $L(\chi, 1) \neq 0$.

Steps. — (I) Let $\zeta_q(s)$ be the function defined for $\Re(s) > 1$ by

$$\zeta_q(s) = \prod_{\chi \pmod q} L(\chi, s) = \sum_{n \geq 1} \frac{a_q(n)}{n^s},$$

and the opposite of its logarithmic derivative

$$-\frac{\zeta_q'(s)}{\zeta_q(s)} = \sum_{n \geq 1} \frac{\Lambda_q(n)}{n^s},$$

Compute $\Lambda_q(n)$ and check that $\Lambda_q(n) \geq 0$.

(II) Let $t \in \mathbb{R}_{\neq 0}$ and let $H_{q,it}(s)$ be the function defined for $\Re(s) > 1$ by

$$H_{q,it}(s) = \zeta_q(s)^3 (\zeta_q(s + it)\zeta_q(s - it))^2 \zeta_q(s + 2it)\zeta_q(s - 2it),$$

and the opposite of its logarithmic derivative

$$-\frac{H_{q,it}'(s)}{H_{q,it}(s)} = \sum_{n \geq 1} \frac{\Lambda_{q,it}(n)}{n^s}.$$

Prove that these two functions extend meromorphically to the open half-plane $\{s \in \mathbb{C}, \Re(s) > 0\}$, and that $\Lambda_{q,it}(n) \geq 0$.

(III) We assume that there exists a Dirichlet character χ such that $L(\chi, 1 + it) = 0$ for some $t \in \mathbb{R}_{\neq 0}$. Check that $a_q(n) \in \mathbb{R}$ for any $n \geq 1$, and deduce that $H_{q,it}(1) = 0$. Prove also that $-H_{q,it}'(s)/H_{q,it}(s)$ has a simple pole at $s = 1$ and that

$$\operatorname{res}_{s=1} \left(-\frac{H_{q,it}'(s)}{H_{q,it}(s)} \right) < 0.$$

(IV) Find a contradiction by examining the Laurent expansion of $-H'_{q,it}(s)/H_{q,it}(s)$ at 1. Conclude.

Exercise 2 **(The Prime Number Theorem in arithmetic progressions)** *Let $q \geq 1$ and $a \in \mathbb{Z}$ coprime to q, and let also*

$$\psi(x; q, a) = \sum_{\substack{n \leq x \\ n = a \pmod q}} \Lambda(n).$$

The goal of this exercise is to prove that the results of Exercise 1 imply

$$\psi(x; q, a) = \frac{x}{\varphi(q)}(1 + o_q(1)).$$

We will also make use of the following lemma.

Lemma. — *Let $f : \mathbb{R}_{\geq 1} \to \mathbb{R}$ be a piecewise continuous function whose absolute value is bounded. Then the function $F(s)$ defined by the integral*

$$F(s) = \int_1^\infty f(t) t^{-s} \, dt$$

is holomorphic in the open half-plane $\{s \in \mathbb{C}, \Re(s) > 1\}$. Moreover, if $F(s)$ extends meromorphically to an open neighborhood of the closed half-plane $\{s \in \mathbb{C}, \Re(s) \geq 1\}$ and if it is holomorphic on the line $\{s \in \mathbb{C}, \Re(s) = 1\}$, then the integral

$$\int_1^\infty f(t) t^{-1} \, dt$$

is convergent and equals $F(1)$.

Steps. — (I) Let $L(\Lambda \cdot \delta_{a \pmod q}, s)$ be the function defined for $\Re(s) > 1$ by

$$L(\Lambda \cdot \delta_{a \pmod q}, s) = \sum_{\substack{n \geq 1 \\ n = a \pmod q}} \frac{\Lambda(n)}{n^s}.$$

Write $L(\Lambda \cdot \delta_{a \pmod q}, s)$ as a linear combination of the logarithmic derivatives $-L'(\chi, s)/L(\chi, s)$ and show that the function

$$L(\Lambda \cdot \delta_{a \pmod q}, s) - \frac{1}{\varphi(q)} \frac{s}{s-1}$$

extends meromorphically to the open half-plane $\{s \in \mathbb{C}, \Re(s) > 0\}$ and that it does not have poles in the closed half-plane $\{s \in \mathbb{C}, \Re(s) \geq 1\}$.

(II) For $t \geq 1$, we set

$$f_{q,a}(t) = \frac{1}{t}\left(\psi(t; q, a) - \frac{t}{\varphi(q)}\right).$$

Prove that $f_{q,a}(t)$ is a piecewise continuous function whose absolute value is bounded.

(III) Prove that for $\Re(s) > 1$, we have

$$L(\Lambda \cdot \delta_{a \pmod q}, s) - \frac{1}{\varphi(q)} \frac{s}{s-1} = -s \int_1^\infty f_{q,a}(t) t^{-s} \mathrm{d}t.$$

(IV) Prove that the integral

$$\int_1^\infty \frac{1}{t^2}\left(\psi(t;q,a) - \frac{t}{\varphi(q)}\right) \mathrm{d}t$$

is convergent and that for every fixed $\lambda > 1$, we have

$$\lim_{x\to\infty} \int_x^{\lambda x} \frac{1}{t^2}\left(\psi(t;q,a) - \frac{t}{\varphi(q)}\right) \mathrm{d}t = 0.$$

(V) Prove that for every $\varepsilon > 0$, there exists $x_0 \geq 1$ such that for every $x \geq x_0$, we have

$$\psi(x;q,a) \leq \left(\frac{1}{\varphi(q)} + \varepsilon\right) x.$$

For this, assume that there exists a sequence $(x_n)_{n\geq 1}$ going to infinity, and such that for every $n \geq 1$,

$$\psi(x_n;q,a) > \left(\frac{1}{\varphi(q)} + \varepsilon\right) x_n,$$

and find a contradiction.

(VI) Conclude.

Exercise 3 **(On the number of zeros of the Riemann zeta function)**

Let $N(T)$ be the number of zeros $s = \sigma + it$ of the Riemann zeta function satisfying $0 < \sigma < 1$ and $0 < t < T$, counted with multiplicities. The goal of this exercise is to prove that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Steps. — (I) Let $\xi(s)$ be the function defined by

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Prove that $2\pi N(T) = \Delta_R \arg\xi(s)$, where R is the rectangle with vertices 2, $2+iT$, $-1+iT$, -1 described in the positive direction, and Δ_R denotes the variation of $\xi(s)$ along the rectangle R.

(II) Using the functional equation satisfied by $\xi(s)$, prove that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + S(T) + \frac{7}{8} + O\left(\frac{1}{T}\right),$$

where $\pi S(T) = \Delta_L \arg\zeta(s)$ and L is the path of line segments joining 2 to $2+iT$ and then to $1/2+iT$.

(III) Prove that

$$\sum_\rho \frac{1}{1+(T-\Im(\rho))^2} \ll \log T,$$

where the sum is over the non-trivial zeros ρ of the Riemann zeta function, counted with multiplicities.

(IV) Let $s=\sigma+it$ with $-1\leq\sigma\leq 2$ and t not equal to an ordinate of a zero. Prove that

$$\frac{\zeta'(s)}{\zeta(s)}=\sum_{\rho}{}' \frac{1}{s-\rho}+O(\log|t|),$$

where the dash on the summation indicates that it is restricted to those ρ for which $|t-\Im(\rho)|<1$.

(V) Use the last step to conclude that $S(T)\ll\log T$.

To go further. — *Using a similar method, one can prove an analog result for Dirichlet L-functions. Let $q\geq 1$ and let χ be a primitive Dirichlet character modulo q. Let also $N(T,\chi)$ be the number of zeros $s=\sigma+it$ of $L(s,\chi)$ satisfying $0<\sigma<1$ and $-T<t<T$, counted with multiplicities. We have*

$$\frac{1}{2}N(T,\chi)=\frac{T}{2\pi}\log\frac{qT}{2\pi}-\frac{T}{2\pi}+O(\log qT).$$

***Exercise 4* (Weil's explicit formula)**. — *Let $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}_{>0})$ and let $\tilde{\varphi}$ be its Mellin transform. We define the function $\check{\varphi}$ by $\check{\varphi}(x)=x^{-1}\varphi(x^{-1})$. We have the identity*

$$\sum_{n\geq 1}(\varphi(n)+\check{\varphi}(n))\,\Lambda(n)=\tilde{\varphi}(1)+\tilde{\varphi}(0)-\sum_{\rho}\tilde{\varphi}(\rho)+\frac{1}{2\pi i}\int_{(1/2)}\left(\frac{\zeta_\infty'(s)}{\zeta_\infty(s)}+\frac{\zeta_\infty'(1-s)}{\zeta_\infty(1-s)}\right)\tilde{\varphi}(s)\,ds,$$

where the sum in the right-hand side is over all non-trivial zeros ρ of the Riemann ζ function, counted with multiplicities, and where

$$\zeta_\infty(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

Steps. — (I) Recall the definition of the function $\xi(s)$ given in Exercise 3. Prove that

$$\frac{1}{2\pi i}\int_{(3/2)}\frac{\xi'(s)}{\xi(s)}\tilde{\varphi}(s)\mathrm{d}s=\frac{1}{2\pi i}\int_{(3/2)}\left(\frac{1}{s}+\frac{1}{s-1}+\frac{\zeta_\infty'(s)}{\zeta_\infty(s)}\right)\tilde{\varphi}(s)\mathrm{d}s-\sum_{n\geq 1}\Lambda(n)\varphi(n).$$

(II) Let $T\geq 1$ and let R_T be the rectangle whose vertices are $3/2\pm iT$ and $-1/2\pm iT$. Prove that

$$\frac{1}{2\pi i}\int_{R_T}\frac{\xi'(s)}{\xi(s)}\tilde{\varphi}(s)\mathrm{d}s=\sum_{\substack{\rho\\ |\Im(\rho)|\leq T}}\tilde{\varphi}(\rho),$$

and also that

$$\sum_{\rho}\tilde{\varphi}(\rho)=\frac{1}{2\pi i}\int_{(3/2)}\frac{\xi'(s)}{\xi(s)}\tilde{\varphi}(s)\mathrm{d}s-\frac{1}{2\pi i}\int_{(-1/2)}\frac{\xi'(s)}{\xi(s)}\tilde{\varphi}(s)\mathrm{d}s.$$

(III) Use the functional equation to prove that

$$\sum_{\rho}\tilde{\varphi}(\rho)=\frac{1}{2\pi i}\int_{(3/2)}\left(\frac{1}{s}+\frac{1}{s-1}+\frac{\zeta_\infty'(s)}{\zeta_\infty(s)}\right)(\tilde{\varphi}(s)+\tilde{\varphi}(1-s))\,\mathrm{d}s-\sum_{n\geq 1}\Lambda(n)\,(\varphi(n)+\check{\varphi}(n)).$$

(IV) Conclude.

Exercise 5 **(Consequence of the Generalized Riemann Hypothesis)**

Let $q \geq 1$ and $a \in \mathbb{Z}$ coprime to q, and recall the definition of $\psi(x; q, a)$ given in Exercise 2. The goal of this exercise is to prove that the Generalized Riemann Hypothesis implies that

$$\psi(x; q, a) - \frac{x}{\varphi(q)} \ll x^{1/2} (\log x)^2,$$

where the constant involved in the notation $\ll$ does not depend on q.

Steps. — (I) Assuming the Riemann Hypothesis, prove that

$$\psi(x) - x \ll x^{1/2} (\log x)^2,$$

where, as usual,

$$\psi(x) = \sum_{n \leq x} \Lambda(n).$$

To do this, use the following approximate explicit formula, which is valid if the distance from x to any prime power is, say, at least $1/2$,

$$\psi(x) = x - \sum_{\substack{\rho \\ |\Im(\rho)| \leq T}} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} + \frac{1}{2} \log\left(1 - x^{-2}\right) + O\left(\frac{x (\log Tx)^2}{T}\right),$$

where the sum is over the non-trivial zeros ρ of the Riemann zeta function, counted with multiplicities.

(II) Let χ be a non-trivial character modulo q. Assuming the Generalized Riemann Hypothesis, prove that

$$\psi(x, \chi) \ll x^{1/2} (\log qx)^2,$$

where, as usual,

$$\psi(x, \chi) = \sum_{n \leq x} \Lambda(n) \chi(n).$$

To do this, use the following approximate explicit formula, which is valid if the distance from x to any prime power is, say, at least $1/2$,

$$\psi(x, \chi) = - \sum_{\substack{\rho \\ |\Im(\rho)| \leq T}} \frac{x^\rho}{\rho} - (1 - \ell)(\log x + b(\chi)) + \sum_{m=1}^{\infty} \frac{x^{\ell - 2m}}{2m - \ell} + O\left(\frac{x (\log qTx)^2}{T}\right),$$

where the sum is over the non-trivial zeros ρ of the L-function $L(s, \chi)$, counted with multiplicities, and $\ell = 0$ if $\chi(-1) = 1$, $\ell = 1$ if $\chi(-1) = -1$, and finally

$$b(\chi) = \lim_{s \to 0} \left(\frac{L'(s, \chi)}{L(s, \chi)} - \frac{1}{s}\right).$$

(III) Conclude.

To go further. — *On average over the residue classes modulo q, one expects much more to be true. Assuming the Generalized Riemann Hypothesis, one can prove that*

$$\sum_{a \pmod q} \left(\psi(x; q, a) - \frac{x}{\varphi(q)}\right)^2 \ll x (\log qx)^4,$$

where the summation is over a coprime to q. What does it imply on the average size of $|\psi(x; q, a) - x/\varphi(q)|$ as a varies?

***Exercise 6* (On the least prime in an arithmetic progression)**
Let $q \geq 1$ and $a \in \mathbb{Z}$ coprime to q. Assuming the Generalized Riemann Hypothesis, prove that there exists a prime number $p = a \pmod{q}$ satisfying $p \ll q^2(\log q)^4$.

***Exercise 7* (On the least quadratic non-residue).** — *Let q be prime and let $n(q)$ be the least quadratic non-residue modulo q. Assuming the Generalized Riemann Hypothesis, prove that $n(q) \ll (\log q)^4$.*

***Exercise 8* (The Class Number Formula for imaginary quadratic fields)**
Let K be an imaginary quadratic field and let $\mathcal{O}_K$ be the ring of integers of K. We let ω_K be the number of roots of unity in K, and $\operatorname{Disc}(K)$ be the discriminant of K. We also let h_K be the class number of K, that is the cardinality of the group

$$\mathcal{C}(K) = I(K)/P(K),$$

where $I(K)$ is the free abelian group generated by prime ideals of $\mathcal{O}_K$, and $P(K)$ is the subgroup of principal ideals. If I is an ideal of $\mathcal{O}_K$, recall that its norm $N(I)$ is defined by

$$N(I) = \#\mathcal{O}_K/I.$$

We define the Dedekind zeta function of the field K by the formula

$$\zeta_K(s) = \sum_{I \subset \mathcal{O}_K} N(I)^{-s},$$

which is valid for $\Re(s) > 1$, and where the sum is over all ideals I of $\mathcal{O}_K$. Prove that $\zeta_K(s)$ has a simple pole at $s = 1$ and that

$$\operatorname{res}_{s=1} \zeta_K(s) = \frac{2\pi h_K}{\omega_K |\operatorname{Disc}(K)|^{1/2}}.$$

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