TOPICS IN NUMBER THEORY - EXERCISE SHEET II

École Polytechnique Fédérale de Lausanne

***Exercise 1* (The Pólya-Vinogradov inequality).** — *Let $q \geq 1$ be an integer and let χ be a non-trivial Dirichlet character modulo q. Let also $H, N \geq 1$. Prove that*

$$\sum_{n=N+1}^{N+H} \chi(n) \ll q^{1/2} \log q.$$

Steps. — (I) Prove that it suffices to deal with the case where χ is a primitive character modulo q.

(II) If χ is a primitive character modulo q, recall that

$$\tau_\chi(n) = \tau_\chi(1)\overline{\chi(n)},$$

where the Gauss sum $\tau_\chi(n)$ is defined by

$$\tau_\chi(n) = \sum_{a \pmod q} \chi(a) e\left(\frac{an}{q}\right),$$

where $e(x) = e^{2\pi i x}$.

***Exercise 2* (On the least quadratic non-residue).** — *Let q be prime and let $n(q)$ be the least quadratic non-residue modulo q. Prove that $n(q) \ll q^{1/2} \log q$.*

To go further. — *Recall that the Generalized Riemann Hypothesis implies that $n(q) \ll (\log q)^4$. Unconditionally, one can prove that $n(q) \ll q^{1/4e^{1/2}+\varepsilon}$ for any fixed $\varepsilon > 0$. What is the bound we obtain if we use Lemma 1 given in Exercise 5?*

***Exercise 3* (The Hadamard three-lines Theorem).** — *Let σ_1, σ_2 be two real numbers such that $\sigma_1 < \sigma_2$. Let also f be a holomorphic function of finite order in the strip $\{z \in \mathbb{C}, \sigma_1 \leq \Re(z) \leq \sigma_2\}$. Suppose that the quantities*

$$M(\sigma) = \sup_{\Re(z)=\sigma} |f(z)|$$

exist for $\sigma = \sigma_1$ and $\sigma = \sigma_2$. Then, for $\sigma_1 \leq \sigma \leq \sigma_2$, $M(\sigma)$ is well-defined. Moreover, $\log M(\sigma)$ is a convex function on $\sigma_1 \leq \sigma \leq \sigma_2$. In other words, for $0 \leq u \leq 1$, we have

$$M(u\sigma_1 + (1-u)\sigma_2) \leq M(\sigma_1)^u M(\sigma_2)^{1-u}.$$

Steps. — (I) By considering the function $z \mapsto mf(z)e^{\lambda z}$ where

$$m = \left(\frac{M(\sigma_2)^{\sigma_1}}{M(\sigma_1)^{\sigma_2}}\right)^{1/(\sigma_2-\sigma_1)},$$

and

$$\lambda = \frac{1}{\sigma_2 - \sigma_1} \log\left(\frac{M(\sigma_1)}{M(\sigma_2)}\right),$$

check that it suffices to prove the result for $M(\sigma_1) = M(\sigma_2) = 1$.

(II) To deal with the case $M(\sigma_1) = M(\sigma_2) = 1$, use the Maximum Modulus Principle.

***To go further* (The Hadamard three-circles Theorem)**

It is easy to check that the following result follows from Exercise 3.

Let r_1, r_2 be two real numbers such that $r_1 < r_2$. Let also f be a holomorphic function in the annulus $\{z \in \mathbb{C}, r_1 \leq |z| \leq r_2\}$. For $r_1 \leq r \leq r_2$, define

$$M(r) = \sup_{|z|=r} |f(z)|.$$

Then $\log M(r)$ is a convex function of $\log r$ on $r_1 \leq r \leq r_2$.

***To go further* (The Phragmén-Lindelöf Principle).** — *In Exercise 4, we will make use of the following generalization of the result of Exercise 3.*

Let σ_1, σ_2 be two real numbers such that $\sigma_1 < \sigma_2$. Let also f be a holomorphic function of finite order in the strip $\{z \in \mathbb{C}, \sigma_1 \leq \Re(z) \leq \sigma_2\}$. Suppose that we have the bounds

$$|f(\sigma_j + it)| \leq M_j(1 + |t|)^{\alpha_j}$$

for $j = 1, 2$, and for some $M_1, M_2 \in \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Then, for $0 \leq u \leq 1$, we have

$$|f(u\sigma_1 + (1-u)\sigma_2 + it)| \leq M_1^u M_2^{1-u}(1 + |t|)^{\alpha_1 u + \alpha_2(1-u)}.$$

***Exercise 4* (Convexity bounds for Dirichlet L-functions)**

Let χ be a Dirichlet character modulo q, and let $\delta, \varepsilon > 0$ be fixed. Prove that we have

$$L(\chi, \sigma + it) \ll 1,$$

uniformly in q and t, in the region $\sigma \geq 1 + \delta$. Prove that this bound implies that

$$L(\chi, \sigma + it) \ll (q(1 + |t|))^{1/2-\sigma},$$

for $\sigma \leq -\delta$. Deduce that

$$L(\chi, \sigma + it) \ll (q(1 + |t|))^{1/2-\sigma/2+\varepsilon},$$

for $0 \leq \sigma \leq 1$.

***To go further* (Terminology).** — *The upper bounds*

$$L(\chi, \sigma + it) \ll (q(1 + |t|))^{1/2-\sigma/2+\varepsilon},$$

in the region $0 \leq \sigma \leq 1$, are called the convexity bounds for Dirichlet L-functions. Any bounds improving upon these are called subconvexity bounds. Exercise 5 gives examples of such bounds.

Exercise 5 **(Subconvexity bounds for Dirichlet L-functions)**
In this exercise, we assume that we know the following result.

Lemma 1. — *Let $\varepsilon > 0$ be fixed. Let $q \geq 1$ be an integer and let χ be a non-trivial Dirichlet character modulo q. Let also $H, N \geq 1$. We have the bound*

$$\sum_{n=N+1}^{N+H} \chi(n) \ll H^{1/2} q^{3/16+\varepsilon}.$$

Let $\varepsilon > 0$ be fixed. Prove that, for any fixed $T > 0$ and $|t| \leq T$, we have the subconvexity bounds

$$L(\chi, \sigma + it) \ll q^{(4-5\sigma)/8+\varepsilon},$$

for $0 \leq \sigma \leq 1/2$, and

$$L(\chi, \sigma + it) \ll q^{(3-3\sigma)/8+\varepsilon},$$

for $1/2 \leq \sigma \leq 1$, where the constants involved in the notation $\ll$ may depend on T.

Pierre Le Boudec - *Fall 2014*
École Polytechnique Fédérale de Lausanne