TOPICS IN NUMBER THEORY - EXERCISE SHEET IV

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

***Exercise 1* (The Brun-Titchmarsh Theorem).** — *Let $x, y > 1$, and let also $1 \leq q < y$ and $a \geq 1$ be coprime to q. Prove that*

$$\pi(x+y;q,a) - \pi(x;q,a) < \frac{2y}{\varphi(q)\log(y/q)} + O\left(\frac{y}{q\log(y/q)^2}\right),$$

where the implied constant is absolute.

***Exercise 2* (On the Hardy-Littlewood Conjecture).** — *Let $k \geq 1$ be fixed and let $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ be such that for any prime p, we have*

$$\{a_i \pmod{p}, i = 1, \ldots, k\} \neq \mathbb{Z}/p\mathbb{Z}.$$

Let $\pi(x;\mathbf{a})$ be the number of $m \leq x$ such that for all $i \in \{1, \ldots, k\}$, $m - a_i$ is prime. Prove that we have

$$\pi(x;\mathbf{a}) \leq 2^k k! C_{\mathbf{a}} \frac{x}{(\log x)^k}\left(1 + O\left(\frac{\log\log x}{\log x}\right)\right),$$

where

$$C_{\mathbf{a}} = \prod_p \left(1 - \frac{\omega(p)}{p}\right)\left(1 - \frac{1}{p}\right)^k,$$

where $\omega(p)$ is the number of roots of the polynomial $(X - a_1)\cdots(X - a_k)$ modulo p.

***To go further* (The Hardy-Littlewood Conjecture).** — *We conjecture that*

$$\pi(x;\mathbf{a}) = C_{\mathbf{a}} \frac{x}{(\log x)^k}(1 + o(1)),$$

so the upper bound of the previous exercise is larger than the expectation by a factor $2^k k!$.

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