TOPICS IN NUMBER THEORY - EXERCISE SHEET I

École Polytechnique Fédérale de Lausanne

***Exercise 1* (Poisson Summation Formula).** — *Let $\mathcal{S}(\mathbb{R})$ be the Schwartz class, that is*

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}, \mathbb{R}), f^{(k)}(x) = O\left((1+|x|)^{-\ell}\right), k, \ell \geq 0 \right\}.$$

For $f \in \mathcal{S}(\mathbb{R})$, we define the Fourier transform $\hat{f}$ of f by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx.$$

Prove that we have the formula

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{m \in \mathbb{Z}} \hat{f}(m).$$

Steps. —

(I) Check that the function F defined for $x \in \mathbb{R}$ by

$$F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$$

is well-defined and is periodic of period 1.

(II) Compute the Fourier coefficients of F and conclude.

***Exercise 2* (On the theta function).** — *Let θ be the function defined for $x > 0$ by*

$$\theta(x) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 x}.$$

Prove that θ satisfies the functional equation

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right),$$

for any $x > 0$.

Steps. —

(I) For fixed $x > 0$, compute the Fourier transform of the function f_x defined for $t \in \mathbb{R}$ by

$$f_x(t) = e^{-\pi x t^2}.$$

(II) Conclude using the Poisson summation formula.

***Exercise 3* (On the cotangent function).** — *Show that, for $z \in \mathbb{C} \smallsetminus \mathbb{Z}$, we have*

$$\pi \cot \pi z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}.$$

Steps. — (I) For fixed $z \in \mathbb{C} \smallsetminus \mathbb{Z}$, prove that for $t \in [0, \pi]$, we have

$$\cos zt = \frac{\sin \pi z}{\pi} \left(\frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{(-1)^n}{z^2 - n^2} \cos nt \right).$$

To do this, compute the Fourier coefficients of the 2π-periodic function ϕ_z defined for $t \in (-\pi, \pi]$ by $\phi_z(t) = \cos zt$.

(II) Conclude.

***Exercise 4* (On the Gamma function).** — *Let Γ be the function defined for $\Re(s) > 0$ by*

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx.$$

Prove that Γ admits a meromorphic continuation to $\mathbb{C}$ with poles only at the non-positive integers, all of order 1. Prove also that, for $n \geq 0$, we have

$$\operatorname{res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}.$$

Steps. — (I) Prove that for $\Re(s) > 0$ and for $m \geq 1$, we have

$$\Gamma(s+m) = \prod_{j=0}^{m-1} (s+j) \Gamma(s).$$

(II) Conclude.

***Exercise 5* (Euler's Reflection Formula).** — *Show that, for $s \in \mathbb{C} \smallsetminus \mathbb{Z}$, we have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Deduce that Γ does not have any zeros in $\mathbb{C}$.

Steps. — (I) Show that, for $0 < \Re(s) < 1$, we have

$$\Gamma(s)\Gamma(1-s) = \int_0^{\infty} \frac{t^{s-1}}{1+t} dt.$$

(II) Deduce that, for $0 < \Re(s) < 1$, we have

$$\Gamma(s)\Gamma(1-s) = \int_0^1 \frac{t^{s-1}}{1+t} dt + \int_0^1 \frac{t^{-s}}{1+t} dt.$$

(III) Prove that, for $0 < \Re(s) < 1$, we have

$$\Gamma(s)\Gamma(1-s) = \frac{1}{s} + 2s \sum_{n=1}^{\infty} \frac{(-1)^n}{s^2 - n^2}$$

(IV) Conclude using the proof of Exercise 3.

To go further. — *We can prove that, for $s \in \mathbb{C}$, we have*

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n},$$

where γ is Euler's constant. Euler's reflection formula thus implies that, for $s \in \mathbb{C}$, we have

$$\frac{\sin \pi s}{\pi s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right),$$

***Exercise 6* (On the Riemann zeta function)**. — *Let ζ be the Riemann zeta function, that is the function defined for $\Re(s) > 1$ by*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

In this exercise, we prove that ζ admits a meromorphic continuation to $\mathbb{C}$, with a single pole at 1, of order 1 and with residue 1.

In addition, we prove that ζ satisfies the functional equation

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-(1-s)/2} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),$$

for $s \in \mathbb{C} \setminus \{0, 1\}$.

Steps. — (I) Prove that for $\Re(s) > 1$, we have

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^{\infty} x^{s/2-1} \left(\sum_{n=1}^{\infty} e^{-\pi n^2 x}\right) dx.$$

(II) Use Exercise 2 to prove that, for $\Re(s) > 1$, we have

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} \left(x^{-(s+1)/2} + x^{s/2-1}\right) \left(\sum_{n=1}^{\infty} e^{-\pi n^2 x}\right) dx.$$

(III) Conclude.

To go further. — *The functional equation shows that ζ has zeros at the negative even integers. The Riemann Hypothesis states that all the other zeros of ζ are on the line $\Re(s) = 1/2$.*

***Exercise 7* (Bernoulli numbers)**. — *For $n \geq 1$, the Bernoulli numbers B_n are defined using the generating series*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

Prove that for $n \geq 0$, $B_n \in \mathbb{Q}$.

Prove also that for $n \geq 1$, we have $B_{2n+1} = 0$. For this, we will show that the function g defined for $x \in \mathbb{R}$ by

$$g(x) = \frac{x}{e^x - 1} + \frac{x}{2}$$

is an even function.

***Exercise 8* (Values of the Riemann zeta function at integers)**

Use Exercise 3 to prove that for $n \geq 1$, we have

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}.$$

Deduce that for $n \geq 0$, we have

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

To go further. — *This implies in particular that $\pi^{-2n}\zeta(2n) \in \mathbb{Q}$ for any $n \geq 1$. We know that $\zeta(3) \notin \mathbb{Q}$. However, it is an open problem whether, for $n \geq 2$, $\zeta(2n+1)$ is a rational number or not.*

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