TOPICS IN NUMBER THEORY - EXERCISE SHEET III

École Polytechnique Fédérale de Lausanne

Exercise 1 **(On α-multiplicative arithmetic functions)**

A function $\mathbb{Z}_{\geq 1} \to \mathbb{C}$ is called an arithmetic function. An arithmetic function g is said to be multiplicative if for $m, n \geq 1$ such that $(m,n) = 1$, we have

$$g(mn) = g(m)g(n).$$

Moreover, g is said to be completely multiplicative if we have $g(mn) = g(m)g(n)$ for all $m, n \geq 1$.

Finally, if α is a non-zero completely multiplicative function then g is said to be α-multiplicative if for $m, n \geq 1$, we have

$$\sum_{d|(m,n)} \alpha(d) g\left(\frac{mn}{d^2}\right) = g(m)g(n).$$

We write $g = 0$ if and only if $g(n) = 0$ for all $n \geq 1$. All along this exercise, α is a fixed non-zero completely multiplicative function.

Prove that any α-multiplicative function is multiplicative.

Let g be a non-zero α-multiplicative function. Show that $g(1) = 1$. Prove also that for $c \in \mathbb{C}$, cg is α-multiplicative if and only if $c = 0$ or $c = 1$.

Prove that if g and h are α-multiplicative then $g + h$ is α-multiplicative if and only if $g = 0$ or $h = 0$.

Let $g_1, \ldots, g_r$ be distinct non-zero α-multiplicative functions. Prove that $g_1, \ldots, g_r$ are linearly independent. Assume now that there exists $(\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ such that the function g defined by

$$g = \sum_{i=1}^{r} \lambda_i g_i$$

is α-multiplicative. Prove that $g = 0$ or $g = g_i$ for some $i \in \{1, \ldots, r\}$.

Let μ be the Möbius function, that is the function defined by $\mu(n) = 0$ if there is a prime number p such that $p^2 \mid n$ and $\mu(n) = (-1)^{\omega(n)}$ if n is squarefree, where $\omega(n)$ denotes the number of prime factors of n. Check that the function μ is multiplicative. Prove also that for $n \geq 1$, we have

$$\sum_{d|n} \mu(d) = \delta_{n=1},$$

where $\delta_{n=1} = 1$ if $n = 1$ and 0 otherwise.

Show that if g is α-multiplicative then for $m, n \geq 1$, we have

$$\sum_{d|n} \mu(d) g(mnd) g\left(\frac{n}{d}\right) = \alpha(n) g(m).$$

Prove also that for any prime number p and any $\ell \geq 0$, we have

$$g(p^{\ell+2}) = g(p) g(p^{\ell+1}) - \alpha(p) g(p^{\ell}).$$

Let $U_n(x)$ be the Chebyshev polynomials of the second kind, that is the polynomials defined by $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$ and, for $\ell \geq 1$,

$$U_{\ell+2}(x) = 2x U_{\ell+1}(x) - U_{\ell}(x).$$

Let g be a non-zero α-multiplicative arithmetic function. Show that for any prime number p and any $\ell \geq 1$, we have

$$g(p^{\ell}) = \alpha(p)^{\ell/2} U_{\ell}\left(\frac{g(p)}{2\alpha(p)^{1/2}}\right).$$

Note that the right-hand side is well-defined if $\alpha(p) = 0$ and that it does not depend on the choice of the square root of $\alpha(p)$.

***Exercise 2* (On the Dirichlet convolution of arithmetic functions)**
Let g_1, g_2 be two arithmetic functions and let $g_1 * g_2$ be their Dirichlet convolution, that is the arithmetic function defined for $n \geq 1$, by

$$(g_1 * g_2)(n) = \sum_{d|n} g_1(d) g_2\left(\frac{n}{d}\right).$$

For $i \in \{1, 2\}$, let

$$G_i(s) = \sum_{n=1}^{\infty} \frac{g_i(n)}{n^s}$$

be the Dirichlet series respectively associated to g_1 and g_2. Assume that there exists $\sigma > 0$ such that the series G_1 and G_2 converge absolutely in the half-plane $\Re(s) > \sigma$.

Prove that for $\Re(s) > \sigma$, we have

$$G_1(s) G_2(s) = \sum_{n=1}^{\infty} \frac{(g_1 * g_2)(n)}{n^s}.$$

***Exercise 3* (On Euler products).** — *Let g be a non-zero arithmetic function. Assume that there exists $\sigma > 0$ such that the Dirichlet series G associated to g converges absolutely in the half-plane $\Re(s) > \sigma$.*

Show that if g is multiplicative then for $\Re(s) > \sigma$, we have

$$G(s) = \prod_{p} \left(\sum_{i=0}^{\infty} \frac{g(p^i)}{p^{is}}\right).$$

Prove that if g is completely multiplicative then for $\Re(s) > \sigma$, we have

$$G(s) = \prod_{p} \left(1 - \frac{g(p)}{p^s}\right)^{-1}.$$

Let α be a non-zero completely multiplicative arithmetic function. Prove that if g is α-multiplicative then for $\Re(s) > \sigma$, we have

$$G(s) = \prod_p \left(1 - \frac{g(p)}{p^s} + \frac{\alpha(p)}{p^{2s}}\right)^{-1}.$$

***Exercise 4* (On the Riemann zeta function).** — *Recall that the Riemann zeta function ζ is defined for $\Re(s) > 1$ by*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Show that for $\Re(s) > 1$, we have

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Prove also that

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

***Exercise 5* (On Dirichlet series associated to Eisenstein series)**

Let $k \geq 2$ and let φ_k be the Dirichlet series associated to the normalized Eisenstein series of weight $2k$. The series φ_k is thus defined for $\Re(s) > 2k$ by

$$\varphi_k(s) = \sum_{n=1}^{\infty} \frac{\sigma_{2k-1}(n)}{n^s},$$

where

$$\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}.$$

Prove that for $\Re(s) > 2k$, we have

$$\varphi_k(s) = \zeta(s)\zeta(s + 1 - 2k).$$

***Exercise 6* (On the Dirichlet series associated to the Discriminant Δ)**

Let φ be the Dirichlet series associated to the normalized modular cusp form of weight 12. The series φ is thus defined for $\Re(s) > 7$ by

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s},$$

where τ is the Ramanujan τ function defined by

$$(2\pi)^{-12}\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n,$$

where $q = e^{2\pi i z}$.

Prove that for $\Re(s) > 7$, we have

$$\varphi(s) = \prod_p \left(1 - \frac{\tau(p)}{p^s} + \frac{1}{p^{2s-11}}\right)^{-1}.$$

Exercise 7 **(On Fourier coefficients of modular forms)**

Prove that for $n \geq 1$, we have

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_3(n-j),$$

and

$$11\sigma_9(n) = 21\sigma_5(n) - 10\sigma_3(n) + 5040 \sum_{j=1}^{n-1} \sigma_3(j)\sigma_5(n-j).$$

Show that for $n \geq 1$, we have

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{691}{3} \sum_{j=1}^{n-1} \sigma_5(j)\sigma_5(n-j).$$

Prove that this implies Ramanujan's congruence

$$\tau(n) = \sigma_{11}(n) \mod 691,$$

for $n \geq 1$.

Exercise 8 **(On the Petersson inner product)**. — *Let M_k^0 denote the space of modular cusp forms of weight $2k$. For $f_1, f_2 \in M_k^0$, we define the Petersson inner product of f_1 and f_2 by*

$$\langle f_1, f_2 \rangle = \int_D f_1(z)\overline{f_2(z)}y^{2k}\frac{dxdy}{y^2},$$

where

$$D = \{z \in \mathbb{H}, |\Re(z)| < 1/2, |z| > 1\}.$$

Prove that for $f_1, f_2 \in M_k^0$, $\langle f_1, f_2 \rangle$ is well-defined.

Let Γ be the modular group. Show that for any $A \in \Gamma$, and for any $f_1, f_2 \in M_k^0$, we have

$$\int_{A \cdot D} f_1(z)\overline{f_2(z)}y^{2k}\frac{dxdy}{y^2} = \int_D f_1(z)\overline{f_2(z)}y^{2k}\frac{dxdy}{y^2},$$

Prove that $\langle \cdot, \cdot \rangle$ is a hermitian inner product.

To go further. — *It is possible to show that the Hecke operators are hermitian with respect to the Petersson inner product. This allows one to prove the existence of an orthonormal basis of M_k^0 consisting of simultaneous eigenforms. In addition, this implies that the eigenvalues of the Hecke operators are real numbers.*

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