Exercise Sheet 1

1. **Dirichlet characters.** A Dirichlet character mod q is an arithmetic function $\chi : \mathbb{Z} \to \mathbb{C}$ which satisfies the following conditions:

i) $\chi(n) = \chi(m) \quad \text{for} \quad n \equiv m \bmod q,$
ii) $\chi(n) \neq 0 \quad \text{exactly when} \quad (n,q) = 1,$
iii) $\chi(n)\chi(m) = \chi(mn) \quad \text{for all} \quad n, m \in \mathbb{Z}.$

The principal character χ_0 mod q is defined as

$$\chi_0(n) = \begin{cases} 1 & \text{if} \quad (n,q) = 1, \\ 0 & \text{if} \quad (n,q) \neq 1. \end{cases}$$

Check that the set of all Dirichlet characters mod q forms a group under pointwise multiplication with identity element χ_0 and inverses given by $\chi^{-1} = \overline{\chi}$, and show that this group is isomorphic to $\mathbb{Z}_q^\times$.

Hint: Use the fundamental theorem of finite abelian groups.

2. Prove the orthogonality relations for Dirichlet characters:

$$\sum_{a \bmod q} \chi(a) = \begin{cases} \varphi(q) & \text{if} \quad \chi = \chi_0, \\ 0 & \text{if} \quad \chi \neq \chi_0, \end{cases} \qquad \text{(for a Dirichlet character } \chi \bmod q)$$

and

$$\sum_{\chi \bmod q} \chi(a) = \begin{cases} \varphi(q) & \text{for} \quad a \equiv 1 \bmod q, \\ 0 & \text{otherwise.} \end{cases}$$

3. **Dirichlet L-functions.** Let χ be a Dirichlet character mod q. The Dirichlet L-function $L(s,\chi)$ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad \text{for} \quad \mathrm{Re}(s) > 1.$$

Prove that $L(s, \chi)$ can be written as an Euler product:

$$L(s, \chi) = \prod_{p \text{ prime}} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad \text{for} \quad \mathrm{Re}(s) > 1$$

What is the relationship between $L(s, \chi_0)$ and $\zeta(s)$?

4. Let $\chi \neq \chi_0$ be a Dirichlet character mod q. Initially, the Dirichlet L-function $L(\chi, s)$ is defined only for $\mathrm{Re}(s) > 1$. Find an analytic continuation of $L(\chi, s)$ to the half-plane $\mathrm{Re}(s) > 0$.

Hint: First prove that

$$\left| \sum_{n \leq x} \chi(n) \right| \leq q \quad \textit{for all} \quad x > 0,$$

and then use partial summation.

5. Let χ be a Dirichlet character. Prove that, for $s \in \mathbb{R}$, $s > 1$,

$$\log L(s, \chi) = \sum_{p \text{ prime}} \frac{\chi(p)}{p^s} + \mathcal{O}(1).$$

6. Let $s \in \mathbb{R}$, $s > 1$. Prove that

$$\prod_{\chi \bmod q} L(s, \chi) \geq 1,$$

where the product runs over all Dirichlet characters mod q.

7. Let the modulus q be fixed. Show that there can be at most one character $\chi \bmod q$ such that $L(1, \chi) = 0$, and that this χ must be real.

8. Let $\chi \neq \chi_0$ be a real Dirichlet character mod q. Define

$$T(x) = \sum_{n \leq x} \frac{\rho(n)}{n^{\frac{1}{2}}} \quad \text{with} \quad \rho(n) = \sum_{d \mid n} \chi(d).$$

Prove that

$$\text{i)} \;\; T(x) \gg \log x \qquad \text{and} \qquad \text{ii)} \;\; T(x) = 2L(1, \chi) x^{\frac{1}{2}} + \mathcal{O}(1).$$

What follows for the value of $L(1, \chi)$?

Hints: For i), show that $\rho(n) \geq 0$ and $\rho(n^2) \geq 1$. For ii), use the hyperbola method and partial summation.

9. Dirichlet's theorem on primes in arithmetic progressions. Let $(a, q) = 1$. Show that the series

$$\sum_{\substack{p \text{ prime} \\ p \equiv a \bmod q}} \frac{1}{p}$$

is divergent. In other words, there are infinitely many primes $p \equiv a \bmod q$.