Exercise Sheet 2

1. **Primitive characters.** Let χ be a Dirichlet character mod q. We say that χ is induced by the character χ_1 mod q_1, if $q_1 \mid q$ and

$$\chi(n) = \chi_1(n) \quad \text{for all} \quad (n, q) = 1.$$

A character is called primitive if it is induced only by itself. Show that any Dirichlet character χ is induced by a unique primitive character χ_1. What is the relationship between $L(s, \chi)$ and $L(s, \chi_1)$?

2. Let $q = rs$ with $r > 1$, and let χ be a primitive Dirichlet character mod q. Prove that

$$S(v) = \sum_{a \bmod r} \chi(as + v)$$

vanishes for any $v \in \mathbb{Z}$.

Hint: The sum $S(v)$ is periodic mod s, and $\chi(b)S(v) = S(bv)$ for $(b, q) = 1$.

3. **Gauss sums.** Let χ be a primitive Dirichlet character mod q. The Gauss sum of χ is defined as

$$\tau(\chi) := \sum_{a \bmod q} \chi(a) e\left(\frac{a}{q}\right).$$

Prove that

$$\tau(\overline{\chi})\chi(n) = \sum_{a \bmod q} \overline{\chi}(a) e\left(\frac{an}{q}\right).$$

Is this statement still true when χ is not primitive?

4. Show that $|\tau(\chi)| = \sqrt{q}$ for any primitive Dirichlet character χ mod q.

Hint: Look at $|\tau(\chi)|^2$ and use Exercise 3.

5. Let χ be a primitive Dirichlet character mod q such that $\chi(-1)=1$. We define

$$\theta(x,\chi)=\sum_{n\in\mathbb{Z}}\chi(n)e^{-\pi n^2\frac{x}{q}}.$$

Check that $\theta(x,\chi)$ is well-defined for $x\in(0,\infty)$ by showing that

$$\theta(x,\chi)\ll\sqrt{\frac{q}{x}}\quad\text{for}\quad x\leq 1,\qquad \theta(x,\chi)\ll e^{-\pi\frac{x}{q}}\quad\text{for}\quad x\geq 1,$$

and prove the functional equation

$$\theta\left(\frac{1}{x},\chi\right)=\frac{\sqrt{xq}}{\tau(\overline{\chi})}\theta(x,\overline{\chi}).$$

Hint: Use Exercise 3 to write $\chi(n)$ in terms of additive characters, so that you can then use Poisson summation.

6. **The functional equation for Dirichlet L-functions (even case).** Let χ be a primitive Dirichlet character mod q such that $\chi(-1)=1$. Show that $L(s,\chi)$ can be continued analytically to the whole complex plane, and prove the functional equation

$$\pi^{-\frac{1-s}{2}}q^{\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)L(1-s,\overline{\chi})=\frac{\sqrt{q}}{\tau(\chi)}\pi^{-\frac{s}{2}}q^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi).$$

What goes wrong when $\chi(-1)=-1$?

Hint: Start with the integral representation

$$2\pi^{-\frac{s}{2}}q^{\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)L(s,\chi)=\int_0^\infty\theta(x,\chi)x^{\frac{s}{2}}\frac{dx}{x},$$

and make use of the functional equation for $\theta(x,\chi)$.

7. **The functional equation for Dirichlet L-functions (odd case).** Let χ be a primitive Dirichlet character mod q such that $\chi(-1)=-1$. Find an analytic continuation of $L(s,\chi)$ to the whole complex plane, and show that it satisfies the functional equation

$$\pi^{-\frac{2-s}{2}}q^{\frac{2-s}{2}}\Gamma\left(\frac{2-s}{2}\right)L(1-s,\overline{\chi})=\frac{i\sqrt{q}}{\tau(\chi)}\pi^{-\frac{s+1}{2}}q^{\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi).$$

Hint: Look at the function

$$\theta_1(x,\chi)=\sum_{n\in\mathbb{Z}}\chi(n)ne^{-\pi n^2\frac{x}{q}},$$

and prove a functional equation relating $\theta_1(1/x,\chi)$ with $\theta_1(x,\overline{\chi})$. Then follow the steps of Exercise 6.