# Exercise Sheet 3

**1. Quadratic Gauss sums.** Let $a$ and $c$ be integers such that $(a,c)=1$. The quadratic Gauss sum $G(a;c)$ is defined as

$$G(a;c) := \sum_{n \bmod c} e\left(\frac{an^2}{c}\right).$$

Check that $G(-a;c) = \overline{G(a;c)}$ and $G(ab^2;c) = G(a;c)$ for $(b,c)=1$, and show that

$$G(a;c_1c_2) = G(c_2a;c_1)G(c_1a;c_2) \quad \text{for} \quad (c_1,c_2)=1.$$

**2. The Legendre symbol.** Let $p$ be an odd prime. The Legendre symbol is defined as

$$\left(\frac{n}{p}\right) := \begin{cases} 1 & \text{if } n \text{ is a quadratic residue mod } p \text{ and } (n,p)=1, \\ -1 & \text{if } n \text{ is a quadratic non-residue mod } p, \\ 0 & \text{if } p \mid n. \end{cases}$$

Show that $\left(\frac{\cdot}{p}\right)$ defines a Dirichlet character mod $p$, and prove that

$$\left(\frac{n}{p}\right) = \frac{G(n;p)}{G(1;p)} \quad \text{for} \quad (n,p)=1.$$

**3.** Show that, for $(a,p)=1$,

$$G(a;p) = \sum_{n \bmod p} \left(\frac{n}{p}\right) e\left(\frac{an}{p}\right).$$

**4. Möbius transformations.** Set

$$\gamma z := \frac{az+b}{cz+d} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(2,\mathbb{R}).$$

Show that these transformations define a group action of $\mathrm{GL}^+(2,\mathbb{R})$ on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

**5**. We set

$$\tilde{\Theta}(z) := \sum_{n\in\mathbb{Z}} e\left(\frac{n^2 z}{2}\right).$$

Check that this defines a holomorphic function $\tilde{\Theta} : \mathbb{H} \to \mathbb{C}$, and show that it satisfies the transformation laws

$$\tilde{\Theta}(T^2 z) = \tilde{\Theta}(z) \quad \text{and} \quad \tilde{\Theta}\left(Wz\right) = \sqrt{-iz}\tilde{\Theta}(z),$$

where $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $W = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

**6**. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix in $\mathrm{SL}(2,\mathbb{Z})$ such that $c > 0$. Prove the following transformation laws for $\tilde{\Theta}(z)$:

$$\Theta(\gamma z) = \frac{G(a; 2c)}{2}\sqrt{\frac{cz+d}{ic}}\tilde{\Theta}(z) \qquad \text{for} \quad c, b \equiv 0 \bmod 2,$$

and

$$\tilde{\Theta}(\gamma z) = G\left(\frac{a}{2}; c\right)\sqrt{\frac{cz+d}{ic}}\tilde{\Theta}(z) \qquad \text{for} \quad a, d \equiv 0 \bmod 2.$$

*Hints: Write* $\gamma z$ *as*

$$\gamma z = \frac{a}{c} - \frac{1}{c(cz+d)}.$$

*It might be helpful to show first that, for* $u \in \mathbb{R}$ *and* $w \in \mathbb{C}$ *with* $\mathrm{Re}(w) > 0$,

$$\sum_{n\in\mathbb{Z}} e^{-\pi w(n+u)^2} = \frac{1}{\sqrt{w}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi n^2}{w} - 2\pi i n u}.$$

**7**. Let $c, d > 0$, such that $c$ is even and $(c, d) = 1$. Prove that

$$\frac{G(-2c; d)}{d^{\frac{1}{2}}} = \frac{G(d; 2c)}{2(ic)^{\frac{1}{2}}}.$$

*Hint: Use the transformation laws for* $\tilde{\Theta}(z)$.

**8**. Let $b, d$ be odd integers, and $c$ an integer coprime to $bd$. Show that

$$\frac{G(c; b^2 d)}{\sqrt{b^2 d}} = \frac{G(c; d)}{\sqrt{d}}.$$

*Hint: Use Exercise 7.*

**9. The Jacobi symbol.** We can extend the Legendre symbol to odd integers by setting

$$\left(\frac{n}{c}\right) := \frac{G(n;c)}{G(1;c)} \quad \text{for} \quad (n,c) = 1 \qquad \text{and} \qquad \left(\frac{n}{c}\right) := 0 \quad \text{for} \quad (n,c) > 1.$$

Let $c = {p_1}^{r_1} \cdots {p_k}^{r_k}$ be the prime factorization of $c$. Prove that

$$\left(\frac{n}{c}\right) = \prod_{i=1}^{k} \left(\frac{n}{p_i}\right)^{r_i},$$

where $\left(\frac{n}{p_i}\right)$ is the Legendre symbol defined above. This shows that $\left(\frac{\cdot}{c}\right)$ defines a Dirichlet character mod $c$.

**10. The law of quadratic reciprocity.** Let $c, d > 1$ be odd coprime integers. Prove that

$$\left(\frac{c}{d}\right)\left(\frac{d}{c}\right) = (-1)^{\frac{c-1}{2}\frac{d-1}{2}}.$$

*Hint: Use Exercise 7 to compute $G(1;c)$.*