Exercise Sheet 4

1. **Gamma function.** Remember that the Gamma function $\Gamma(s)$ is defined as

$$\Gamma(s) := \int_0^\infty e^{-t} t^s \, \frac{dt}{t} \quad \text{for} \quad \mathrm{Re}(s) > 0.$$

Show that $\Gamma(s)$ can be continued meromorphically to the whole complex plane by

$$\Gamma(s) = \int_1^\infty e^{-t} t^s \, \frac{dt}{t} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(s+n)},$$

and show that it satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s).$$

2. Prove that

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\dots(x+n)} \quad \text{for} \quad x > 0.$$

Hint: Show first that

$$\int_0^1 t^{x-1}(1-t)^n \, dt = \frac{n!}{x(x+1)\dots(x+n)}.$$

3. Prove that

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}} \quad \text{for} \quad s \in \mathbb{C},$$

where γ is the Euler-Mascheroni constant.

4. **Euler's reflection formula.** Prove that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Hint: Use the formula

$$\frac{\sin \pi s}{\pi s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right),$$

which follows from the Hadamard factorization theorem.

5. **Legendre duplication formula.** Prove that

$$\Gamma(s)\Gamma\left(s + \tfrac{1}{2}\right) = \sqrt{\pi} 2^{1-2s} \Gamma(2s).$$

6. **Stirling's formula for $n!$.** Show that the factorial of n can be approximated by

$$n! \sim \sqrt{2\pi n} \frac{n^n}{e^n} \quad \text{as} \quad n \to \infty.$$

Hint: We have

$$n! = \sqrt{n} \frac{n^n}{e^n} \int_{-\sqrt{n}}^{\infty} \left(1 + \frac{t}{\sqrt{n}}\right)^n e^{-t\sqrt{n}} \, dt.$$

7. Prove that

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log(2\pi) + \int_0^{\infty} \frac{\lfloor u \rfloor - u + \frac{1}{2}}{u + s} \, du.$$

Hint: Start by explicitly evaluating the integral

$$\int_0^N \frac{\lfloor u \rfloor - u + \frac{1}{2}}{u + s} \, du.$$

8. **Stirling's formula for $\Gamma(s)$.** Let $\varepsilon > 0$. Show that $\Gamma(s)$ can be approximated by

$$\Gamma(s) = \sqrt{2\pi} s^{\left(s - \frac{1}{2}\right)} e^{-s} \left(1 + \mathcal{O}\left(\frac{1}{|s|}\right)\right) \quad \text{for} \quad |\arg s| \leq \pi - \varepsilon.$$

9. Let $\varepsilon > 0$. Show that

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + \mathcal{O}\left(\frac{1}{|s|^2}\right) \quad \text{for} \quad |\arg s| \leq \pi - \varepsilon.$$

Hint: Use Cauchy's integral formula.

10. Let σ_0, σ_1 be real numbers. Show that

$$|\Gamma(\sigma + it)| = \sqrt{2\pi} |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|} \left(1 + \mathcal{O}\left(\frac{1}{|t|}\right)\right)$$

for $\sigma_0 \leq \sigma \leq \sigma_1$ and $|t| \geq 1$.