Exercise Sheet 5

1. **Completed Dirichlet L-functions.** Let χ be a primitive Dirichlet character mod q. The completed Dirichlet L-function $\Lambda(s,\chi)$ is defined as

$$\Lambda(s,\chi) := q^{\frac{s}{2}}\gamma(s,\chi)L(s,\chi),$$

where

$$\gamma(s,\chi) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s+\kappa}{2}\right) \quad \text{and} \quad \kappa := \begin{cases} 0 & \text{if} \quad \chi(-1)=1, \\ 1 & \text{if} \quad \chi(-1)=-1. \end{cases}$$

Show that $\Lambda(s,\chi)$ is a function of order 1.

2. Show that

$$-\frac{L'}{L}(s,\chi) = \frac{\log q}{2} + \frac{\gamma'}{\gamma}(s,\chi) - B(\chi) - \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right),$$

where $B(\chi)$ is a complex number such that

$$\mathrm{Re}(B(\chi)) = -\sum_\rho \mathrm{Re}(\rho^{-1}).$$

Here the sums over ρ run over all zeros of $L(s,\chi)$ inside the critical strip counted with their multiplicity.

3. Let $\mathcal{N}(T,\chi)$ be the number of zeros of $L(s,\chi)$ inside the critical strip which have imaginary part in $[-T,T]$ counted with multiplicity. Prove that

$$\mathcal{N}(T+1,\chi) - \mathcal{N}(T,\chi) \ll \log(q(T+2)) \quad \text{and} \quad \mathcal{N}(T,\chi) \ll T\log(q(T+2)).$$

Hint: Show first that

$$\sum_\rho \frac{1}{1+|T-\mathrm{Im}(\rho)|^2} \ll \log(q(T+2)).$$

4. Prove that, for $-\frac{1}{2} \leq \mathrm{Re}(s) \leq 2$ and $|\mathrm{Im}(s)| \geq 1$, we can approximate $\frac{L'}{L}(s,\chi)$ by

$$\frac{L'}{L}(s,\chi) = \sum_{\substack{\rho \\ |\mathrm{Im}(\rho - s)| \leq 1}} \frac{1}{s-\rho} + \mathcal{O}\left(\log(q|\mathrm{Im}(s)|)\right).$$

5. **Weil's explicit formula for Dirichlet L-functions.** Let $f : (0,\infty) \to \mathbb{C}$ be a smooth and compactly supported function, and let $\tilde{f}$ be its Mellin inverse. Prove the identity

$$\sum_n \Lambda(n) \left(\chi(n) f(n) + \overline{\chi}(n) n^{-1} f(n^{-1})\right)$$
$$= f(1)\log q + \frac{1}{2\pi i}\int_{(1/2)} \left(\frac{\gamma'}{\gamma}(s,\chi) + \frac{\gamma'}{\gamma}(1-s,\overline{\chi})\right) \tilde{f}(s)\, ds - \sum_\rho \tilde{f}(\rho).$$

6. Let $X, Y \in \mathbb{R}$ such that $1 \leq Y \leq X$. Construct a smooth and compactly supported function $f : (0,\infty) \to \mathbb{R}$ which satisfies

$$f(\xi) = 1 \quad \text{for} \quad \xi \in \left[\tfrac{X}{2}, X\right] \quad \text{and} \quad \operatorname{supp} f \subset \left[\tfrac{X-Y}{2}, X+Y\right],$$

and

$$f^{(\nu)}(\xi) \ll Y^{-\nu} \quad \text{for} \quad \nu \geq 0.$$

7. Let f be the function constructed above. Show that its Mellin transform satisfies the bound

$$\tilde{f}(s) \ll X^{\mathrm{Re}(s)} \min\left(1, \frac{X}{Y|s||s+1|}\right).$$

8. Prove that

$$\sum_{\frac{x}{2} < n \leq x} \Lambda(n) = \frac{x}{2} + \mathcal{O}\left(x e^{-C\sqrt{\log x}}\right),$$

for some constant $C > 0$.

Hint: Use Weil's explicit formula for the Riemann zeta function with the weight function constructed in Exercise 6.

9. **Prime number theorem.** Prove that

$$\sum_{n \leq x} \Lambda(n) = x + \mathcal{O}\left(x e^{-C\sqrt{\log x}}\right),$$

for some constant $C > 0$.