Exercise Sheet 6

1. Given an L-function $L(f,s)$, let

$$N(f,T) := \#\left\{\rho \in \mathbb{C} \,|\, L(f,\rho) = 0,\ 0 \le \mathrm{Re}(\rho) \le 1,\ |\,\mathrm{Im}(\rho)| \le T\right\}.$$

Assuming $L(f,s)$ has no zeros with $|\,\mathrm{Im}(\rho)| = T$, prove that

$$N(f,T) = \mathrm{Re}\left(\frac{1}{\pi i}\int_C \frac{\Lambda'}{\Lambda}(f,s)\,ds\right) + \mathcal{O}\left(\log(2q(f))\right),$$

where C is the path starting at $\frac{1}{2} - iT$, passing through the points $3 - iT$ and $3 + iT$, and ending at $\frac{1}{2} + iT$.

Hint: The implied constants here and below may depend on the degree d and the local parameters $\mu_{f,i}$ at infinity of $L(f,s)$.

2. Show that, for $T \ge 2$,

$$\mathrm{Re}\left(\frac{1}{\pi i}\int_C \frac{\Gamma'}{\Gamma}\left(\frac{s+\mu_{f,i}}{2}\right)\,ds\right) = \frac{2T}{\pi}\log\frac{T}{2e} + \mathcal{O}\left(\log T\right).$$

3. Show that, for $T \ge 2$,

$$\mathrm{Re}\left(\frac{1}{\pi i}\int_C \frac{L'}{L}(f,s)\,ds\right) \ll \log(q(f)T).$$

4. **Counting zeros of $L(f,s)$.** Prove the following asymptotic formula for the number of zeros of $L(f,s)$,

$$N(f,T) = \frac{T}{\pi}\log\frac{q(f)T^d}{(2\pi e)^d} + \mathcal{O}\left(\log(q(f)T)\right).$$