Heights of CM points I
Gross–Zagier formula

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1 Introduction and state of main results

In 1984, Gross and Zagier [GZ] proved a formula that relates the Neron–Tate heights of Heegner points to the central derivatives of some Rankin L-series under certain ramification conditions. Since then some generalizations are given in various papers [Zh1, Zh2, Zh3]. The methods of proofs of the Gross–Zagier theorem and all its extensions depend on some newform theories. There are essential difficulties to remove all ramification assumptions in this method. The aim of this paper is a proof of a general formula in which all ramification condition are removed. Such a formula is an analogue of a central value formula of Waldspurger [Wa] and has been more or less formulated by Gross in 2002 in term of representation theory. In the following, we want to describe statements of the main results and main idea of proof.

1.1 L-function and root numbers

Let F be a number field with adele ring $\mathbb{A} = \mathbb{A}_F$. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$. Let E be a quadratic extension of F, and $\chi : E^\times \backslash \mathbb{A}_E^\times \to \mathbb{C}^\times$ be a character of finite order. We assume that

$$\chi|_{\mathbb{A}^\times} \cdot \omega_\pi = 1$$

where ω_π is the central character of π.

Denote by $L(s, \pi, \chi)$ the Rankin-Selberg L-function. Denote by $\epsilon(\frac{1}{2}, \pi_v, \chi_v) = \pm 1$ the local root number at each place v of F, and denote

$$\Sigma = \{v : \epsilon(1/2, \pi_v, \chi_v) \neq \chi_v \eta_v(-1)\},$$

where η_v is the quadratic character associated to the extension E_v/F_v. Then Σ is a finite set and the global root number is given by

$$\epsilon(1/2, \pi, \chi) = \prod_v \epsilon(1/2, \pi_v, \chi_v) = (-1)^{\#\Sigma}.$$

The following is a description of root numbers in terms of linear functionals:

Proposition 1.1.1 (Saito–Tunnell [Tu, Sa]). *Let v be place of F and B_v a quaternion division algebra over F_v. Let π'_v be the Jacquet–Langlands correspondence of π_v on $B_v^\times$ if π_v is square integrable. Fix embeddings $E_v^\times \subset \mathrm{GL}_2(F_v)$ and $E_v^\times \subset B_v^\times$ as algebraic subgroups. Then $v \in \Sigma$ if and only if*

$$\mathrm{Hom}_{E_v^\times}(\pi_v \otimes \chi_v, \mathbb{C}) = 0.$$

Moreover

$$\dim \mathrm{Hom}_{E_v^\times}(\pi_v \otimes \chi_v, \mathbb{C}) + \mathrm{Hom}_{E_v^\times}(\pi'_v \otimes \chi_v, \mathbb{C}) = 1.$$

Here the second space is treated as 0 if π'_v is not square integrable.

Arithmetic properties of the L-function depends heavily on the parity of $\#\Sigma$. When $\#\Sigma$ is even, the central value $L(\frac{1}{2}, \pi, \chi)$ is related to certain period integral. Explicit formulae have been given by Gross, Waldspurger and S. Zhang. We will recall the treatment of Waldspurger [Wa] in next section.

When $\#\Sigma$ is odd then $L(\frac{1}{2}, \pi, \chi) = 0$. Under the assumption that E/F is a CM-extension, that π_v is discrete of weight 2 for all infinite place v, and that χ is of finite order, then the central derivative $L'(\frac{1}{2}, \pi, \chi)$ is related to the height pairings of some CM divisors on certain Shimura curves. Explicit formulae have been obtained by Gross-Zagier and one of the authors under some unramified assumptions. The goal of this paper is to get a general explicit formula in this odd case without any unramified assumption.

1.2 Waldspurger's formula

Assume that the order of Σ is even in this subsection. We introduce the following notations:

1. B is the unique quaternion algebra over F with ramification set Σ;
2. $B^\times$ is viewed as an algebraic group over F;
3. $T = E^\times$ is a torus of G for a fixed embedding $E \subset B$;
4. $\pi' = \otimes_v \pi'_v$ is the Jacquet–Langlands correspondence of π on $B^\times(\mathbb{A})$.

Define a period integral $\ell(\cdot, \chi) : \pi' \to \mathbb{C}$ by

$$\ell(f, \chi) = \int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} f(t)\chi(t)dt, \quad f \in \pi'.$$

Here the integral uses the Tamagawa measure.

Assume that ω_π is unitary. Then π' is unitary with Petersson inner product $\langle \cdot, \cdot \rangle$ using Tamagawa measure which has volume 2 on $\mathbb{A}^\times B^\times \backslash B^\times(\mathbb{A})$. Fix any non-trivial Hermitian form $\langle \cdot, \cdot \rangle_v$ on π'_v so that their product gives $\langle \cdot, \cdot \rangle$. Waldspurger proved the following formula when ω_π is trivial:

Theorem 1.2.1 (Walspurger when $\omega_\pi = 1$)**.** *Assume that $f = \otimes_v f_v \in \pi'$ is decomposable and nonzero. Then*

$$|\ell(f, \chi)|^2 = \frac{\zeta_F(2) L(\frac{1}{2}, \pi, \chi)}{2\ L(1, \pi, \mathrm{ad})} \prod_{v \le \infty} \alpha(f_v, \chi_v),$$

where

$$\alpha(f_v, \chi_v) = \frac{L(1, \eta_v) L(1, \pi_v, \mathrm{ad})}{\zeta_v(2) L(\frac{1}{2}, \pi_v, \chi_v)} \int_{F_v^\times \backslash E_v^\times} \langle \pi'_v(t) f_v, f_v \rangle_v \chi_v(t) dt.$$

Moreover, $\alpha(f_v, \chi_v)$ is nonzero and equal to 1 for all but finitely many places v.

We interpret the formula as a result on bilinear functionals. As π' is unitary, the contragredient $\widetilde{\pi}'$ is equal to $\bar{\pi}'$. Thus we have two bilinear functionals on $\pi' \otimes \widetilde{\pi}'$. The first one is

$$\ell(f_1, f_2) = \ell(f_1, \chi)\ell(f_2, \chi^{-1}) = \int_{(Z(\mathbb{A})T(F)\backslash T(\mathbb{A}))^2} f_1(t) f_2(t) \chi(t_1) \bar{\chi}(t_2) dt_1 dt_2, \quad f_1 \in \pi', f_2 \in \widetilde{\pi}'.$$

And the second one is the product of local linear functionals:

$$\alpha(f_1, f_2) = \prod_v \alpha(f_{1v}, f_{2v})$$

$$\alpha(f_{1v}, f_{2v}) = \frac{L(1, \eta_v) L(1, \pi_v, \mathrm{ad})}{\zeta_v(2) L(\frac{1}{2}, \pi_v, \chi_v)} \int_{F_v^\times \backslash E_v^\times} \langle \pi'_v(t) f_{1,v}, f_{2,v} \rangle_v \chi_v(t) dt.$$

It is easy to see that both pairings are bilinear and (χ^{-1}, χ)-equivariant under the action of $T(\mathbb{A}) \times T(\mathbb{A})$. But we know such functionals are unique up to scalar multiples by the uniqueness theorem of the local linear functionals of Saito–Tunnell (Proposition 1.1.1). Therefore, these two functionals must be proportional. Theorem 1.1 says that their ratio is recognized as a combination of special values of L-functions.

1.3 Gross–Zagier formula

Now assume that Σ is odd. We further assume that

1. F is totally real and E is totally imaginary.
2. π_v is discrete of weight 2 at all infinite places v of F.
3. χ is a character of finite order.

In this case, the set Σ must contain all infinite places. We have a totally definite quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ which does not have a model over F. For each open compact subgroup U of $\mathbb{B}_f^\times := (\mathbb{B} \otimes_{\mathbb{A}} \mathbb{A}_f)^\times$, we have a Shimura curve X_U whose complex points at a real place τ of F can be described as

$$X_{U,\tau}(\mathbb{C}) = B(\tau)^\times \backslash \mathscr{H}^\pm \times \mathbb{B}_f^\times / U_\infty U$$

where $B(\tau)$ is a quaternion algebra over F with ramification set $\Sigma \backslash \{\tau\}$, $\mathbb{B}_f$ is identified with $B(\tau)_{\mathbb{A}_f}$ as an $\mathbb{A}_f$-algebra, and $B(\tau)^\times$ acts on $\mathscr{H}^\pm$ through an isomorphism $B(\tau)_\tau \simeq M_2(\mathbb{R})$.

Fix an embedding $E_{\mathbb{A}} \longrightarrow \mathbb{B}$, then we have a set $C_U \subset X_U(E^{\mathrm{ab}})$ of CM-points by E which has a (non-canonical) identification:

$$C_U \simeq E^\times \backslash E^\times(\mathbb{A}_f)/U \cap E^\times(\mathbb{A}_f) \tag{1.3.1}$$

with Galois action of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ given by left multiplication of $E^\times(\mathbb{A}_f)$ and the class field theory. More precisely, at a complex place τ_E of E over τ of F, $C_{\tau_E}(\mathbb{C})$ in $X_{U,\tau}(\mathbb{C})$ is represented by (z_0, t) with $t \in E^\times(\mathbb{A}_f)$ and $z_0 \in \mathscr{H}^\pm$ is the unique point fixed by $E^\times$ such that the action of $E^\times$ on the tangent space $T_{\mathscr{H}^\pm, z_0}$ is given by inclusion τ_E.

Assume that $U \cap E^\times(\mathbb{A}_f)$ is included into $\ker \chi$. Define a divisor of degree zero

$$Y_U = \sum_{t \in E^\times \backslash E^\times(\mathbb{A}_f)/U \cap E^\times(\mathbb{A}_f)} \chi(t)([t] - \xi_t)$$

where $[t]$ denote the CM-point in C_U via identification (1.3.1) and ξ_t is the Hodge class of degree 1 in the fiber of X_U containing $[z_0, t]$. The divisor Y_U up to a multiple by a root of unity does not depend on the choice of identification (1.3.1).

Let π'_f denote the Jacquet–Langlands correspondence of π_f on $\mathbb{B}_f^\times$. Let $f \in \pi'_f, \widetilde{f} \in \widetilde{\pi}'_f$. We then have a Hecke operator $\mathrm{T}_{f \otimes \widetilde{f}}$ attached to a $\phi \in C_0^\infty(\mathbb{B}_f^\times)$ which has image $f \otimes \widetilde{f}$ in $\pi'_f \otimes \widetilde{\pi}'_f$ via the projection

$$C_0^\infty(\mathbb{B}_f^\times) \longrightarrow \pi'_f \otimes \widetilde{\pi}'_f$$

and image 0 in the projections $\sigma'_f \otimes \widetilde{\sigma}'_f$ for σ'_f the Jacquet–Langlands correspondences of finite parts of all other cuspidal representations σ of $\mathrm{GL}_2(\mathbb{A})$ with discrete components of weight 2 at all archimedean places. It is easy to shown that the Neron–Tate height $\langle Y, \mathrm{T}_{f \otimes \widetilde{f}} Y \rangle_{NT}$ does not depend on the choice of the lifting ϕ.

Theorem 1.3.1. *Assume that $f = \otimes f_v \in \pi'_f$ and $\widetilde{f} = \otimes \widetilde{f}_v \in \widetilde{\pi}'_f$ are decomposable. Then*

$$\langle Y, \mathrm{T}_{f \otimes \widetilde{f}} Y \rangle_{NT} = \frac{\zeta_F(2) L'(1/2, \pi, \chi)}{4L(1, \pi, ad)} \prod_{v < \infty} \alpha(f_v, \widetilde{f}_v).$$

Remark. For new form f, some partial results have been proved in [Zh1, Zh2, Zh3] with more precise formula under some unramified assumptions.

1.4 Applications

Let π and χ satisfy the same condition as in §1.3. Then we have an abelian variety A defined over E such that

$$L(s, A) = \prod_{\sigma} L(s - \frac{1}{2}, \pi^{\sigma}, \chi^{\sigma})$$

where $(\pi^{\sigma}, \chi^{\sigma})$ are conjugates of (π, χ) for automorphisms σ of $\mathbb{C}$ in the sense that the Heck eigenvalues of π^{σ} and χ^{σ} are σ-conjugates of those of π and χ. Let $\mathbb{Q}[\pi, \chi]$ denote the subfield generated by Heck eigenvalues of π and values of χ. Then A has a multiplication by an order in $\mathbb{Q}[\pi, \chi]$. Replace A by an isogenous one, we may assume that A has multiplication by $\mathbb{Z}[\pi, \chi]$.

Theorem 1.4.1 (Tian–Zhang)**.** *Under the assumption above, we have:*

1. *If* $\mathrm{ord}_{s=1/2} L(s, \pi, \chi) = 1$*, then the Mordell-Weil group* $A(E)$ *as a* $\mathbb{Z}[\pi, \chi]$*-module has rank* 1 *and the Shafarevich-Tate group* Ш(A) *is finite.*

2. *If* $\mathrm{ord}_{s=1/2} L(s, \pi, \chi) = 0$ *and* A *does not have CM-type, then the Mordell-Weil group* $A(E)$ *is finite. Furthermore, if* χ *is trivial, and* A *is geometrically simple, then the Shafarevich-Tate group* Ш(A) *is also finite.*

Remark. For π as above, there is an abelian variety A_π defined over F with L-series given by

$$L(s, A_\pi) = \prod_{\sigma} L(s - 1/2, \pi^{\sigma}).$$

We may assume that A_π has multiplications by the ring $\mathbb{Z}[\pi]$ of the Hecke eigenvalues in π. The variety A up to an isogeny can be obtained by the algebraic tensor product:

$$A := A_\pi \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi, \chi].$$

In terms of algebraic points,

$$A(\bar{E}) = A_\pi(\bar{E}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi, \chi]$$

with usual Galois action by $\mathrm{Gal}(\bar{E}/E)$ on $A(\bar{E})$ and by character χ on $\mathbb{Z}[\pi, \chi]$.

Example. The theorem applies to all modular F-elliptic curves: the elliptic curve A over a Galois extension of F such that A^{σ} is isogenous to A for all $\sigma \in \mathrm{Gal}(\bar{F}/F)$.

1.5 Plan of proof

The ideal of proof of Theorem 1.3.1 which is still based on Gross–Zagier's paper [GZ] is to show that the kernel functions representing the the central derivative of L-series is equal to the generating series of height pairings of CM-points. The new idea here is to construct the kernel function and generating series systematically using Weil representations. This is a variation of an idea used by Waldspurger in the central value formula [Wa].

First of all, for given Σ, we will have a quaternion algebra $\mathbb{B}$ over $\mathbb{A}$ with ramification set Σ. The reduced norm on $\mathbb{B}$ defines an orthogonal space $\mathbb{V}$ with group GO of orthogonal similitudes. Then we have a Weil representation of $\mathrm{GL}_2(\mathbb{A})\times\mathrm{GO}(\mathbb{A})$ on the space $\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$. For each $\phi\in\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$ we will construct an automorphic form $I(s,g,\phi)$ as a mixed Eisenstein and theta series to represent the L-series $L(s,\pi,\chi)$.

If Σ is even, then $\mathbb{B}=B_\mathbb{A}$ for some quaternion algebra B over F. Then by Siegel–Weil formulae, $I(0,g,\chi,\phi)$ is equal to a period integral $\theta(g,\phi,\chi)$ of a theta series associate to ϕ. This is essentially the Waldspurger formula.

If Σ is odd, $I(0,g,\chi,\phi)=0$ and $I'(0,g,\chi,\phi)$ will represent $L'(1/2,\pi,\chi)$. In the case where E/F is a CM-quadratic extension, where π has weight 2 at each archimedean place, and where χ is a finite character, we will write $I'(0,g,\chi)$ as a sum $I'(0,g,\chi,\phi)(v)$ indexed over places v of F and a finite sum of Eisenstein series and their derivations defined in §4.4.4 in our previous paper [Zh1].

The role of theta series in Waldspurger's work is played by the following generating series of heights of CM-points:

$$Z_\phi(g,\chi)=\langle Z_\phi(g)Y_U,Y_U\rangle$$

where $Z_\phi(g)$ is a generating series of Hecke operators on X_U defined by Kudla extending the classic formula $\sum_n T_nq^n$. The modularity of this generating series is proved in our previous paper [YZZ]. Using Arakelov theory, we may decompose $Z_\phi(g,\chi)$ into a sum of pairings involving the Hodge class ξ's and pairings only involving CM-points. The pairings involves ξ's give a linear combination of Eisenstein series and their derivations. The pairing over CM-points can be further decomposed into a sum $Z_\phi(g,\chi)(v)$ indexed over places of F.

For good v, one can show that $I'(0,g,\chi)(v)$ is essentially $Z_\phi(g,\chi)(v)$. For bad v's, we can show that they are essential equal to a pseudo-theta series associate to quadratic space over F. Since both $I'(0,g,\chi,\phi)$ and $Z_\phi(g,\chi)$ are both automorphic, some density argument can be used to show that their difference is essentially a linear combination of theta series and Eisenstein series. Now the main theorem follows from the result of Saito and Tunnel on linear forms, Proposition 1.1.1.

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1.6 Notation and terminology

Local fields and global fields

We normalize the absolute values, additive characters, and measures following Tate's thesis. Let k be a local field of characteristic zero.

- Normalize the absolute value $|\cdot|$ on k as follows:

  It is the usual one if $k=\mathbb{R}$.

It is the square of the usual one if $k = \mathbb{C}$.

It takes the uniformizer to the inverse of the cardinality of the residue field if k is non-archimedean.

- Normalize the additive character $\psi : k \to \mathbb{C}^\times$ as follows:

  If $k = \mathbb{R}$, then $\psi(x) = e^{2\pi i x}$.

  If $k = \mathbb{C}$, then $\psi(x) = e^{4\pi i \mathrm{Re}(x)}$.

  If k is non-archimedean, then it is a finite extension of $\mathbb{Q}_p$ for some prime p. Take $\psi = \psi_{\mathbb{Q}_p} \circ \mathrm{tr}_{\mathrm{k}/\mathbb{Q}_\mathrm{p}}$. Here the additive character $\psi_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is defined by $\psi_{\mathbb{Q}_p}(x) = e^{-2\pi i \iota(x)}$, where $\iota : \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the natural embedding.

- We take the Haar measure dx on k to be self-dual with respect to ψ. More precisely,

  If $k = \mathbb{R}$, then dx is the usual Lebesgue measure.

  If $k = \mathbb{C}$, then dx is twice of the usual Lebesgue measure.

  If k is non-archimedean, then $\mathrm{vol}(O_k) = |d_k|^{\frac{1}{2}}$. Here O_k is the ring of integers and $d_k \in k$ is the different of k over $\mathbb{Q}_p$.

- We take the Haar measure $d^\times x$ on $k^\times$ as follows:

$$d^\times x = \zeta_k(1)|x|_v^{-1} dx.$$

  Recall that $\zeta_k(s) = (1 - N_v^{-s})^{-1}$ if v is non-archimedean with residue field with N_v-elements, and $\zeta_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$, $\zeta_\mathbb{C}(s) = 2(2\pi)^{-s}\Gamma(s)$. With this normalization, if k is non-archimedean, then $\mathrm{vol}(O_k^\times) = \mathrm{vol}(O_k)$.

Now go back to the totally real field F. For each place v, we choose $|\cdot|_v, \psi_v, dx_v, d^\times x_v$ as above. By tensor products, they induce global $|\cdot|, \psi, dx, d^\times x$.

Quadratic extensions

As for the CM field E, we view $V_1 = (E, q)$ as a two-dimensional vector space over F, which uniquely determines self-dual measures dx on $\mathbb{A}_E$ and E_v for each place v of F. We define the measures on the corresponding multiplicative groups by the same setting as above.

Let $E^1 = \{y \in E^\times : q(y) = 1\}$ act on E by multiplication. It induces an isomorphism $\mathrm{SO}(V_1) \simeq E^1$ of algebraic groups over F. We also have $\mathrm{SO}(V_1) = E^\times/F^\times$ given by

$$E^\times/F^\times \to E^1, \quad t \mapsto t/\bar{t}.$$

It is an isomorphism by Hilbert Theorem 90.

We have the following exact sequence

$$1 \to E^1 \to E^\times \xrightarrow{q} q(E^\times) \to 1.$$

For all places v, we endow E_v^1 the measure such that the quotient measure over $q(E_v^\times)$ is the subgroup measure of $F_v^\times$. On the other hand, we endow $E_v^\times/F_v^\times$ with the quotient measure. It turns out that these two measures induce the same one on $\mathrm{SO}(V_1)$.

Let v be a non-archimedean place of F, denote by $d_v \in O_{F_v}$ the local different of F_v, and by $D_v \in O_{F_v}$ the local discriminant of E_v/F_v. Then $\mathrm{vol}(O_{F_v}) = \mathrm{vol}(O_{F_v}^\times) = |d_v|_v^{\frac{1}{2}}$ and $\mathrm{vol}(O_{E_v}) = \mathrm{vol}(O_{E_v}^\times) = |D_v|^{\frac{1}{2}}|d_v|_v$. We set d_v and D_v to be 1 when v is archimedean.

If v is a non-archimedean place inert in E, then

$$\mathrm{vol}(E_v^1) = \frac{\mathrm{vol}(O_{E_v}^\times)}{\mathrm{vol}(q(O_{E_v}^\times))} = \frac{\mathrm{vol}(O_{E_v}^\times)}{\mathrm{vol}(O_{F_v}^\times)} = |D_v|^{\frac{1}{2}}|d_v|^{\frac{1}{2}} = |d_v|^{\frac{1}{2}}.$$

If v is a non-archimedean place ramified in E, then

$$\mathrm{vol}(E_v^1) = \frac{\mathrm{vol}(O_{E_v}^\times)}{\mathrm{vol}(q(O_{E_v}^\times))} = 2\frac{\mathrm{vol}(O_{E_v}^\times)}{\mathrm{vol}(O_{F_v}^\times)} = 2|D_v|^{\frac{1}{2}}|d_v|^{\frac{1}{2}}.$$

If v is archimedean, then $\mathrm{vol}(E_v^1) = 2$.

Notation on GL_2

We introduce the matrix notation:

$$m(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \quad d(a) = \begin{pmatrix} 1 & \\ & a \end{pmatrix}, \quad d^*(a) = \begin{pmatrix} a & \\ & 1 \end{pmatrix}$$
$$n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

We denote by $P \subset \mathrm{GL}_2$ and $B \subset \mathrm{SL}_2$ the subgroups of upper triangular matrices. And N be the standard unipotent subgroup.

For any place v, the character $\delta_v : P(F_v) \to \mathbb{R}^\times$ defined by

$$\delta_v : \begin{pmatrix} a & b \\ & d \end{pmatrix} \mapsto \left|\frac{a}{d}\right|^{\frac{1}{2}}$$

which extends to a function $\delta_v : \mathrm{GL}_2(F_v) \to \mathbb{R}^\times$ by Iwasawa decomposition.

If v is a real place, we define a function $\lambda_v : \mathrm{GL}_2(F_v) \to \mathbb{C}$ by $\lambda_v(g) = e^{i\theta}$ if

$$h = \begin{pmatrix} c & \\ & c \end{pmatrix}\begin{pmatrix} a & b \\ & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

in the form of Iwasawa decomposition.

For $g \in \mathrm{GL}_2(\mathbb{A})$, we define $\delta(g) = \prod_v \delta_v(g_v)$ and $\lambda_\infty(g) = \prod_{v|\infty} \lambda_v(g_v)$.

2 Weil representation and kernel functions

In this section, we will review the theory of Weil representation and its applications to integral representations of Rankin–Selberg L-series $L(s, \pi, \chi)$ and to a proof of Waldspurger's the central value formula. We will mostly follow Waldspurger's treatment with some modifications including the construction of incoherent Eisenstein series from Weil representation.

We will start with the classical theory of Weil representation of $O(F) \times \mathrm{SL}_2(F)$ on $\mathscr{S}(V)$ for an orthogonal space V over a local field F and its extension to $\mathrm{GO}(F) \times \mathrm{GL}_2(F)$ on $\mathscr{S}(V, F)$ by Waldspurger. We then define theta function, state Siegel–Weil formulae, and define normalized local Shimizu lifting. The main result of this section is an integral formula for L-series $L(s, \pi, \chi)$ using a kernel function $I(s, g, \phi, \chi)$. This kernel function is a mixed Eisenstein and theta series attached to each $\phi \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)$ for $\mathbb{V}$ an orthogonal space obtained form a quaternion algebra over $\mathbb{A}$. The Waldspurger formula is a direct consequence of the Siegel–Weil formula. We conclude the section by proving the vanishing of the kernel function at $s = 0$ using information on Whittaker function of incoherent Eisenstein series.

2.1 Weil representations

Let us start with some basic set up on Weil representation. We follow closely from Waldspurger's paper [Wa] but we will work on a slightly bigger space $\mathscr{S}(V, F^\times)$.

Non-archimedean case

Let F be a non-archimedean local field and (V, q) a quadratic space over F. Let $O = O(V, q)$ denote the orthogonal group of (V, q) and $\widetilde{\mathrm{SL}}_2$ the double cover of SL_2. Then for any non-trivial character ψ of F, the group $\widetilde{\mathrm{SL}}_2(F) \times O(F)$ has an action r on the space $\mathscr{S}(V)$ of locally constant with compact support as follows:

- $r(h)\phi(x) = \phi(h^{-1}x), \qquad h \in O(F);$
- $r(m(a))\phi(x) = \chi_V(a)\phi(ax)|a|^{\dim V/2}, \qquad a \in F^\times;$
- $r(n(b))\phi(x) = \phi(x)\psi(bq(x)), \qquad b \in F;$
- $r(w)\phi = \gamma\widehat{\phi}(x), \qquad \gamma^8 = 1.$

Archimedean case

When $F \simeq \mathbb{R}$, we may define an analogous representation of the pair $(\mathscr{G}, \mathscr{K})$ of the Lie algebra $\mathscr{G}$ and a maximal compact subgroup $\mathscr{K} = \widetilde{\mathrm{SO}}_2(\mathbb{R}) \times \mathscr{K}^0$ of $\widetilde{\mathrm{SL}}_2(\mathbb{R}) \times O(\mathbb{R})$. More precisely, the maximal compact subgroup $\mathscr{K}^0$ stabilizes a unique orthogonal decomposition $V = V^+ + V^-$ such that the restrictions of q on $V^\pm$ are positive and negative definite respectively. Then we take $\mathscr{S}(V)$ as to be the space of functions of the form

$$P(x)e^{-2\pi(q(x^+)-q(x^-))}, \qquad x = x^+ + x^-,\ x^\pm \in V^\pm$$

where P is a polynomial function on V. The action of $(\mathscr{G}, \mathscr{K})$ on $\mathscr{S}(V)$ can be deduced formally from the same formulae as above. Notice that the space $\mathscr{S}(V)$ depends on the choice of the maximal compact group $\mathscr{K}^0$ of $O(V)$ and ψ.

Extension to GL_2

Assume that $\dim V$ is even, we want to extend this action to an action r of $\mathrm{GL}_2(F) \times \mathrm{GO}(F)$ or more precisely their $(\mathscr{G}, \mathscr{K})$ analogue in real case. If F is non-archimedean, let $\mathscr{S}(V, F^\times)$ be the space of smooth functions $\phi(x, u)$ on $V \times F^\times$ which is in $\mathscr{S}(V)$ for any fixed u. Then the Weil representation is defined by the following formulae:

- $r(h)\phi(x, u) = \phi(h^{-1}x, \nu(h)u), \qquad h \in \mathrm{GO}(F);$
- $r(g)\phi(x, u) = r_u(g)\phi(x, u), \qquad g \in \mathrm{SL}_2(F);$
- $r(d(a))\phi(x, u) = \phi(x, a^{-1}u)|a|^{-\dim V/4}, \qquad a \in F^\times$

where in the second formula $\phi(x, u)$ is a function of x, and r_u is the Weil representation on V with new norm uq.

Remark. In Waldspurger's paper [Wa], he only considers the Weil representation on the the space $\mathscr{S}(V \times F^\times)$ consisting of locally constant functions with compact support if F is non-archimedean, and linear combinations of functions of the form

$$h(u)P(x)e^{-2\pi|u|(q(x^+)-q(x^-))}$$

with $h(u) \in C_0^\infty(\mathbb{R})$ if F is archimedean.

2.2 Theta series

Now we assume that F is a totally real number field and (V, q) is an orthogonal space over F. Then we can define a Weil representation r on $\mathscr{S}(V_\mathbb{A})$ (which actually depends on ψ) of $\widetilde{\mathrm{SL}}_2(\mathbb{A}) \times O(V_\mathbb{A})$. When $\dim V$ is even, we can define an action r of $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(V_\mathbb{A})$ on $\mathscr{S}(V_\mathbb{A}, \mathbb{A}^\times)$ which is the restricted tensor product of $\mathscr{S}(V_v, F_v)$ with base element as characteristic function of $V_{O_{F_v}} \times O_{F_v}^\times$.

Notice that the representation r depends only on the quadratic space $(V_\mathbb{A}, q)$ over $\mathbb{A}$. Thus we may define representations directly for a pair $(\mathbb{V}, q)$ of a free $\mathbb{A}$-module $\mathbb{V}$ with non-degenerate quadratic form q.

If $(\mathbb{V}, q)$ is a base change of an orthogonal space over F, then we call this Weil representation is *coherent*; otherwise it is called *incoherent*.

Let F be a number field, and (V, q) a quadratic space over F. Fix a nontrivial additive character ψ of $F\backslash\mathbb{A}$. Then for any $\phi \in \mathscr{S}(V_\mathbb{A})$, we can form a theta series as a function on $\mathrm{SL}_2(F)\backslash\widetilde{\mathrm{SL}}_2(\mathbb{A}) \times O(F)\backslash O(\mathbb{A})$:

$$\theta_\phi(g, h) = \sum_{x \in V} r(g, h)\phi(x), \qquad (g, h) \in \widetilde{\mathrm{SL}}_2(\mathbb{A}) \times O(\mathbb{A}).$$

Similarly, when V has even dimension we can define theta series for $\phi \in \mathscr{S}(V_{\mathbb{A}} \times \mathbb{A}^\times)$ as an automorphic form on $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(F)\backslash\mathrm{GO}(\mathbb{A})$:

$$\theta_\phi(g,h) = \sum_{(x,u)\in V\times F^\times} r(g,h)\phi(x,u).$$

Siegel–Weil formulae

Still in the global case. Let V be an orthogonal space over F. For $\phi \in \mathscr{S}(V_{\mathbb{A}})$, $s \in \mathbb{C}$, we have a section

$$g \mapsto \delta(g)^s r(g)\phi(0)$$

in $\mathrm{Ind}_B^{\mathrm{SL}_2}(\chi_V|\cdot|^{\dim V/2+s})$ where δ is the modulo function which assigns g to $|a|^s$ in the Iwasawa decomposition $g = m(a)n(b)k$. Thus we can form an Eisenstein series

$$E(s,g,\phi) = \sum_{\gamma\in P(F)\backslash\mathrm{SL}_2(F)} \delta(\gamma g)^s r(\gamma g)\phi(0).$$

This Then we have Siegel–Weil formula in the following cases:

A. $(V,q) = (E, \mathrm{N}_{E/F})$ with E a quadratic extension of F, and

B. V is the space of the trace free elements in a quaternion algebra over F and q is the the restriction of the reduced norm.

Theorem 2.2.1 (Siegel–Weil formula)**.** *Let $\phi \in \mathscr{S}(V_{\mathbb{A}})$. Then*

$$\int_{\mathrm{SO}(F)\backslash\mathrm{SO}(\mathbb{A})} \theta_\phi(g,h)dh = mE(0,g,\phi)$$

where dh is the Tamagawa measure on $\mathrm{SO}(\mathbb{A})$ which has values $2L(1,\chi)$ and 2 on $\mathrm{SO}(F)\backslash\mathrm{SO}(\mathbb{A})$ for $\dim V = 2$ and 3 respectively, and $m = L(1,\eta)$ and 2 respectively.

Notice that the Eisenstein series can be defined for any quadratic space $(\mathbb{V}, q)$ with rational $\det V$. But its value vanishes on $s = 0$. Kudla has proposed a connection between the derivative of Eisenstein series and arithmetic intersection of certain arithmetic cycles on Shimura varieties.

2.3 Shimizu lifting

Let F be a local field and B a quaternion algebra over F. Write $V = B$ as an orthogonal space with quadratic form q defined by the reduced norm on B. Let $B^\times \times B^\times$ act on V by

$$x \mapsto h_1 x h_2^{-1}, \qquad x \in B, \quad h_i \in B^\times.$$

Then we have an exact sequence:

$$1 \longrightarrow F^\times \longrightarrow (B^\times \times B^\times) \rtimes \{1,\iota\} \longrightarrow \mathrm{GO}(F) \longrightarrow 1.$$

Here ι acts on $V = B$ by the canonical involution, and on $B^\times \times B^\times$ by $(h_1, h_2) \mapsto (h_2^{\iota-1}, h_1^{\iota-1})$, and here $F^\times$ is embedded into the middle group by $x \mapsto (x, x) \rtimes 1$. The theta lifting of any representation π of $\mathrm{GL}_2(F)$ on GO is induced by representation $\mathrm{JL}(\widetilde{\pi}) \otimes \mathrm{JL}(\pi)$ on $B^\times \times B^\times$, $\mathrm{JL}(\pi)$ is the Jacquet–Langlands lifting of π. Recall that $\mathrm{JL}(\pi) \neq 0$ only if $B = M_2(F)$ or π is discrete, and that if $\mathrm{JL}(\pi) \neq 0$,

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(F) \times B^\times \times B^\times}(\mathscr{S}(V \times F^\times) \otimes \pi, \mathrm{JL}(\pi) \otimes \mathrm{JL}(\widetilde{\pi})) = 1.$$

This space has a normalized form θ as follows: for any $\varphi \in \pi$ realized as $W_{-1}(g)$ in a Whittaker model for the additive character ψ^{-1}, $\phi \in \mathscr{S}(V \times F^\times)$, and for $\mathscr{F}$ the canonical form on $\mathrm{JL}(\pi) \otimes \mathrm{JL}(\widetilde{\pi})$,

$$\mathscr{F}\theta(\phi \otimes \varphi) = \frac{\zeta(2)}{L(1, \pi, \mathrm{ad})} \int_{N(F) \backslash \mathrm{GL}_2(F)} W_{-1}(g) r(g) \phi(1, 1) dg \tag{2.3.1}$$

The constant before the integral is used to normalize the form so that $\mathscr{F}(\theta_\phi^\varphi) = 1$ when every thing is unramified.

Let F be a number field and B a quaternion algebra over F. Then the Shimuzu lifting can be realized as a global theta lifting:

$$\theta(\phi \otimes \varphi)(h) = \frac{\zeta(2)}{2L(1, \pi, \mathrm{ad})} \int_{\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g) \theta_\phi(g, h) dg, \qquad h \in B_\mathbb{A}^\times \times B_\mathbb{A}^\times. \tag{2.3.2}$$

Here is the relation between global theta lifting and normalized local theta lifting:

Proposition 2.3.1. *We have a decomposition $\theta = \otimes \theta_v$ in*

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}) \times B_\mathbb{A}^\times \times B_\mathbb{A}^\times}(\mathscr{S}(V_\mathbb{A} \times \mathbb{A}^\times) \otimes \pi, \mathrm{JL}(\widetilde{\pi}) \otimes \mathrm{JL}(\pi)).$$

Proof. It suffices to show that for decomposable $\phi = \otimes \phi_v$ and $\varphi = \otimes \varphi_v$,

$$\mathscr{F}\theta(\phi \otimes \varphi) = \prod \mathscr{F}\theta_v(\phi_v \otimes \varphi_v).$$

The inner product between $\mathrm{JL}(\pi)$ and $\mathrm{JL}(\widetilde{\pi})$ is given by integration on the diagonal $(\mathbb{A}^\times B^\times \backslash B_\mathbb{A}^\times)^2$. Let $V = V_0 \oplus V_1$ correspond to the decomposition $B = B_0 \oplus F$ with B_0 the subspace of trace free elements. Then the diagonal can be identify with $\mathrm{SO}' = \mathrm{SO}(V_0)$ thus we have globally

$$\mathscr{F}\theta(\phi \otimes \varphi) = \int_{\mathrm{SO}'(F) \backslash \mathrm{SO}'(\mathbb{A})} dh \int_{\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A})} \theta_\phi(g, h) \varphi(g) dg.$$

To compute this integral, we interchange the order of integrals and apply the Siegel–Weil formula:

$$\int_{\mathrm{SO}'(F) \backslash \mathrm{SO}'(\mathbb{A})} \theta_\phi(g, h) dh = 2 \left(\sum_{\gamma \in P(F) \backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x,u) \in V_1 \times F} r(\gamma g) \phi(x, u) \right)_{s=0}.$$

Thus we have

$$\begin{aligned}\mathscr{F}\theta(\phi\otimes\varphi) =&2\int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})}\varphi(g)\sum_{\gamma\in P(F)\backslash\mathrm{GL}_2(F)}\sum_{(x,u)\in V_1\times F}r(\gamma g)\phi(x,u)dg\\ =&2\int_{P(F)\backslash\mathrm{GL}_2(\mathbb{A})}\varphi(g)\sum_{(x,u)\in V_1\times F^\times}r(g)\phi(x,u)dg\end{aligned}$$

The sum here can be written as a sum of two parts I_1+I_2 invariant under $P(F)$ where I_1 is the sum over $x\neq 0$ and I_2 over $x=0$. It is easy to see that I_2 is invariant under $N(\mathbb{A})$ which contributes 0 to the integral as φ is cuspidal. The sum I_1 is a single orbit over diagonal group. Thus we have

$$\begin{aligned}&2\int_{N(F)\backslash\mathrm{GL}_2(\mathbb{A})}\varphi(g)\sum_{(x,u)\in V_1\times F}r(g)\phi(1,1)dg\\ =&2\int_{N(\mathbb{A})\backslash\mathrm{GL}_2(\mathbb{A})}W_{-1}(g)r(g)\phi(1,1)dg.\end{aligned}$$

□

2.4 Integral representations of *L*-series

In the following we want to describe an integral representation of the L-series $L(s,\pi,\chi)$. Let F be a number field with ring of adeles $\mathbb{A}$. Let $\mathbb{B}$ be a quaternion algebra with ramification set Σ. Fix an embedding $E_{\mathbb{A}}\longrightarrow\mathbb{B}$. We have an orthogonal decomposition

$$\mathbb{B}=E_{\mathbb{A}}+E_{\mathbb{A}}\mathfrak{j},\qquad \mathfrak{j}^2\in\mathbb{A}^\times.$$

Write $\mathbb{V}$ for the orthogonal space $\mathbb{B}$ with reduced norm q, and $\mathbb{V}_1=E_{\mathbb{A}}$ and $\mathbb{V}_2=E_{\mathbb{A}}\mathfrak{j}$ as subspaces of $V_{\mathbb{A}}$. Then $\mathbb{V}_1$ is coherent and is the base change of F-space $V_1:=E$, and $\mathbb{V}_2$ is coherent if and only if Σ is even.

For $\phi\in\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$, we can form a mixed Eisenstein–Theta series:

$$I(s,g,\phi)=\sum_{\gamma\in P(F)\backslash\mathrm{GL}_2(F)}\delta(\gamma g)^s\sum_{(x_1,u)\in V_1\times F^\times}r(\gamma g)\phi(x_1,u)$$

$$I(s,g,\phi,\chi)=\int_{T(F)\backslash T(\mathbb{A})}\chi(t)I(s,g,r(t,1)\phi)dt.$$

For $\varphi\in\pi$, we want to compute the integral

$$P(s,\varphi,\phi,\chi)=\int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})}\varphi(g)I(s,g,\phi,\chi)dg.$$

Proposition 2.4.1 (Waldspurger). *If $\phi = \otimes\phi_v$ and $\varphi = \otimes\varphi_v$ are decomposable, then*

$$P(s, \varphi, \phi, \chi) = \prod_v P_v(s, \varphi_v, \phi_v, \chi_v)$$

where

$$P_v(s, \varphi_v, \phi_v, \chi_v) = \int_{Z(F_v)\backslash T(F_v)} \chi(t)dt \int_{N(F_v)\backslash \mathrm{GL}_2(F_v)} \delta(g)^s W_{-1,v}(g) r(g)\phi_v(t^{-1}, q(t))dg.$$

Proof. Bring the definition formula of $I(s, g, \phi, \chi)$ to obtain an expression for $P(s, \varphi, \phi, \chi)$:

$$\int_{Z(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \varphi(g)\delta(g)^s \int_{T(F)\backslash T(\mathbb{A})} \chi(t) \sum_{(x,u)\in V_1\times F^\times} r(g, (t, 1))\phi(x, u)dtdg.$$

We decompose the first integral as a double integral:

$$\int_{Z(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} dg = \int_{Z(\mathbb{A})N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} dndg$$

and perform the integral on $N(F)\backslash N(\mathbb{A})$ to obtain:

$$\int_{Z(\mathbb{A})N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s dg \int_{T(F)\backslash T(\mathbb{A})} \chi(t) \sum_{(x,u)\in V_1\times F^\times} W_{q(x)^{-1}u}(g) r(g, (t, 1))\phi(x, u)dt.$$

Here as φ is cuspidal, the term $x = 0$ has no contribution to the integral. In this way, we may change variable $(x, u) \mapsto (x, q(x^{-1})u)$ to obtain the following expression of the sum:

$$\begin{aligned} &\sum_{(x,u)\in E^\times\times F^\times} W_{-u}(g) r(g, (t, 1))\phi(x, q(x^{-1})u) \\ = &\sum_{(x,u)\in E^\times\times F^\times} W_{-u}(g) r(g, (tx, 1))\phi(1, u)). \end{aligned}$$

The sum over $x \in E^\times$ collapses with quotient $T(F) = E^\times$. Thus the integral becomes

$$\int_{Z(\mathbb{A})N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s dg \int_{T(\mathbb{A})} \chi(t) \sum_{u\in F^\times} W_{-u}(g) r(g, (t, 1))\phi(1, u)dt.$$

The expression does not change if we make the substitution $(g, au) \mapsto (gd(a)^{-1}, u)$. Thus we have

$$\int_{Z(\mathbb{A})N(\mathbb{A})P(F)\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s dg \int_{T(\mathbb{A})} \chi(t) \sum_{u\in F^\times} W_{-u}(d(u^{-1})g) r(d(u^{-1})g, (t, 1))\phi(1, u)dt.$$

The sum over $u \in F^\times$ collapses with quotient $P(F)$, thus we obtain the following expression:

$$P(s, \varphi, \phi, \chi) = \int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s dg \int_{T(\mathbb{A})} \chi(t) W_{-1}(g) r(g)\phi(t^{-1}, q(t))dt.$$

We may decompose inside integral as

$$\int_{Z(\mathbb{A})\backslash T(\mathbb{A})}\int_{Z(\mathbb{A})}$$

and move the integral $\int_{Z(\mathbb{A})\backslash T(\mathbb{A})}$ to the outside. Then we use the fact that $\omega_\pi \cdot \chi|_{\mathbb{A}^\times} = 1$ to obtain

$$P(s,\varphi,\phi,\chi) = \int_{Z(\mathbb{A})\backslash T(\mathbb{A})} \chi(t)dt \int_{N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} \delta(g)^s W_{-1}(g) r(g)\phi(t^{-1}, q(t))dg.$$

□

When everything is unramified, Waldspurger has computed these integrals:

$$P_v(s,\varphi_v,\phi_v,\chi_v) = \frac{L((s+1)/2,\pi_v,\chi_v)}{L(s+1,\eta_v)}.$$

Thus we may define a normalized integral P_v^0 by

$$P_v(s,\varphi_v,\phi_v,\chi_v) = \frac{L((s+1)/2,\pi_v,\chi_v)}{L(s+1,\eta_v)} P_v^0(s,\varphi_v,\phi_v,\chi_v).$$

This normalized local integral P_v^0 will be regular at $s=0$ and equal to

$$\frac{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})}{\zeta_v(2)L(1/2,\pi_v,\chi_v)} \int_{Z(F_v)\backslash T(F_v)} \chi_v(t)\mathscr{F}\left(\mathrm{JL}(\pi)(t)\theta(\phi_v\otimes\varphi_v)\right)dt.$$

We may write this as $\alpha_v(\theta(\phi_v\otimes\varphi_v))$ with

$$\alpha_v \in \mathrm{Hom}(\mathrm{JL}(\pi)\otimes\mathrm{JL}(\widetilde{\pi}),\mathbb{C})$$

given by the integration of matrix coefficients:

$$\alpha_v(\varphi\otimes\widetilde{\varphi}) = \frac{L(1,\eta_v)L(1,\pi_v,\mathrm{ad})}{\zeta_v(2)L(1/2,\pi_v,\chi_v)} \int_{Z(F_v)\backslash T(F_v)} \chi_v(t)(\pi(t)\varphi,\widetilde{\varphi})dt. \tag{2.4.1}$$

And we define an element $\alpha := \otimes_v\alpha_v$ in $\mathrm{Hom}(\mathrm{JL}(\pi)\otimes\mathrm{JL}(\widetilde{\pi}),\mathbb{C})$.

We now take value or derivative at $s=0$ to obtain

Proposition 2.4.2.

$$P(0,\varphi,\phi,\chi) = \frac{L(1/2,\pi,\chi)}{L(1,\chi_E)}\prod_v \alpha_v(\theta(\phi_v\otimes\varphi_v)).$$

If Σ is odd, then $L(1/2,\pi,\chi)=0$, and

$$P'(0,\varphi,\phi,\chi) = \frac{L'(1/2,\pi,\chi)}{2L(1,\chi_E)}\alpha(\theta(\phi\otimes\varphi)).$$

Remark. Let $\mathscr{A}_\Sigma(\mathrm{GL}_2,\chi)$ denote the direct sum of cusp forms π on $\mathrm{GL}_2(\mathbb{A})$ such that $\Sigma(\pi,\chi)=\Sigma$. If Σ is even, let $\mathscr{I}(g,\phi,\chi)$ be the projection of $I(0,g,\phi,\chi)$ on $\mathscr{A}_\Sigma(\mathrm{GL}_2,\chi^{-1})$. If Σ is odd, let $\mathscr{I}'(g,\phi,\chi)$ denote the projection of $I'(0,g,\phi,\chi)$ on $\mathscr{A}_\Sigma(\mathrm{GL}_2,\chi^{-1})$. Then have shown that $\mathscr{I}(g,\phi,\chi)$ and $\mathscr{I}'(g,\phi,\chi)$ represent the functionals

$$\varphi\mapsto\frac{L(1/2,\pi,\chi)}{L(1,\chi_E)}\alpha(\widetilde{\theta}(\phi\otimes\varphi))\qquad\text{if }\Sigma\text{ is even}$$

$$\varphi\mapsto\frac{L'(1/2,\pi,\chi)}{2L(1,\chi_E)}\alpha(\widetilde{\theta}(\phi\otimes\varphi))\qquad\text{if }\Sigma\text{ is odd}$$

on π respectively.

2.5 Proof of Waldspurger formula

Assume that Σ is even and we recall Waldspurger's proof of his central value formula. Now the space $\mathbb{V}=V(\mathbb{A})$ is coming from a global $V=B$ over F. For $\phi\in\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$ and $\varphi\in\pi$, we want to compute the double period integral of $\theta(\phi\otimes\varphi)$:

$$\int_{(T(F)Z(\mathbb{A})\backslash T(\mathbb{A}))^2}\theta(\phi\otimes\varphi)(t_1,t_2)\chi(t_1)\chi^{-1}(t_2)dt_1dt_2.$$

Using definition, this equals

$$\frac{\zeta(2)}{2L(1,\pi,\mathrm{ad})}\int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A})}\varphi(g)\theta(g,\phi,\chi)dg$$

where

$$\theta(g,\phi,\chi)=\int_{Z^\Delta(\mathbb{A})T(F)^2\backslash T(\mathbb{A})^2}\theta_\phi(g,(t_1,t_2))\chi(t_1t_2^{-1})dt_1dt_2,$$

where Z^Δ is the image of the diagonal embedding $F^\times\longrightarrow(B^\times)^2$. We change variable $t_1=tt_2$ to get a double integral

$$\theta(g,\phi,\chi)=\int_{T(F)\backslash T(\mathbb{A})}\chi(t)dt\int_{Z(\mathbb{F})T(F)\backslash T(\mathbb{A})}\theta_\phi(g,(tt_2,t_2))dt_2,$$

Notice that the diagonal the diagonal embedding $Z\backslash T$ can be realized as $\mathrm{SO}(Ej)$ in the decomposition $V=B=E+E_j$. Thus we can apply Siegel Weil formula 2.2.1 to obtain

$$\theta(g,\phi,\chi)=L(1,\chi)I(0,g,\phi,\chi).$$

Here recall

$$I(s,g,\phi)=\sum_{\gamma\in P(F)\backslash\mathrm{GL}_2(F)}\delta(\gamma g)^s\sum_{(x,\psi)\in E\times F^\times}r(\gamma g,(t,1)\phi)(x,u).$$

Combining with Proposition 2.4.2, we have

$$\int_{(T(F)Z(\mathbb{A})\backslash T(\mathbb{A}))^2} \theta(\phi\otimes\varphi)(t_1,t_2)\chi(t_1)\chi^{-1}(t_2)dt_1dt_2 = \frac{\zeta(2)L(1/2,\pi,\chi)}{2L(1,\pi,\mathrm{ad})}\alpha(\theta(\phi\otimes\varphi)).$$

This is certainly an identity as functionals on $\mathrm{JL}(\pi)\otimes\mathrm{JL}(\widetilde{\pi})$:

$$\ell_\chi\otimes\ell_{\chi^{-1}} = \frac{\zeta(2)L(1/2,\pi,\chi)}{2L(1,\pi,\mathrm{ad})}\cdot\alpha. \tag{2.5.1}$$

2.6 Vanishing of the kernel function

We want to compute the Fourier expansion of $I(s,g,\phi,\chi)$ and $I(s,g,\phi)$ in the case that Σ is odd, and prove that they vanish at $s=0$. We will mainly work on $\phi\in\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$.

By writing ϕ as a linear combination, we may reduce the computation to the decomposable case: $\phi(x_1+x_2,u)=\phi_1(x_1,u)\phi_2(x_2,u)$ for $x_i\in\mathbb{V}_i$. Then we have

$$I(s,g,\phi)=\sum_{u\in F^\times}\theta(g,u,\phi_1)E(s,g,u,\phi_2)$$

and

$$I(s,g,\phi,\chi)=\int_{T(F)\backslash T(\mathbb{A})}\chi(t)I(s,g,r(t,1)\phi)dt$$

where

$$\theta(g,u,\phi_1)=\sum_{x\in E}r(g)\phi_1(x,u),$$

$$E(s,g,u,\phi_2)=\sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)}\delta(\gamma g)^s r(\gamma g)\phi_2(0,u).$$

It suffices to study the behavior of Eisenstein series at $s=0$. The work of Kudla-Rallis in more general setting shows that the incoherent Eisenstein series $E(s,g,u,\phi_2)$ vanishes at $s=0$ by local reasons. For readers' convenience, we will show this fact by detailed analysis of the local Whittaker functions. We will omit ϕ_1 and ϕ_2 in the notation.

We start with the Fourier expansion of Eisenstein series:

$$E(s,g,u)=E_0(s,g,u)+\sum_{a\in F^\times}E_a(s,g,u)$$

where the a-th Fourier coefficient is

$$E_a(s,g,u)=\int_{F\backslash\mathbb{A}}E\left(s,\begin{pmatrix}1&b\\&1\end{pmatrix}g,u\right)\psi(-ab)db,\quad a\in F.$$

An easy calculation gives the following result.

Lemma 2.6.1.

$$\begin{aligned} E_0(s,g,u) &= \delta(g)^s r_2(g)\phi_2(0,u) - \int_{\mathbb{A}} \delta(wn(b)g)^s \int_{\mathbb{V}_2} r_2(g)\phi_2(x_2,u)\psi(buq(x_2))d_u x_2 db, \\ E_a(s,g,u) &= -\int_{\mathbb{A}} \delta(wn(b)g)^s \int_{\mathbb{V}_2} r_2(g)\phi_2(x_2,u)\psi(b(uq(x_2)-a))d_u x_2 db, \quad a \in F^\times. \end{aligned}$$

Here the measure $d_u x_2$ on $\mathbb{V}_2$ is the self-dual Haar measure with respect to uq.

Proof. It is a standard result that the constant term is given by

$$E_0(s,g,u) = \delta(g)^s r_2(g)\phi_2(0,u) + \int_{N(\mathbb{A})} \delta(wng)^s r_2(wng)\phi_2(0,u)dn.$$

By definition, we have

$$\begin{aligned} r_2(wn(b)g)\phi_2(0,u) &= \gamma(\mathbb{V}_2) \int_{\mathbb{V}_2} r_2(n(b)g)\phi_2(x_2,u)d_u x_2 \\ &= \gamma(\mathbb{V}_2) \int_{\mathbb{V}_2} r_2(g)\phi_2(x_2,u)\psi(buq(x_2))d_u x_2. \end{aligned}$$

Here $\gamma(\mathbb{V}_2)$ is the Weil index of the quadratic space $(\mathbb{V}_2, uq)$, which is apparently independent of $u \in F^\times$. By the orthogonal decomposition $\mathbb{V} = \mathbb{V}_1 \oplus \mathbb{V}_2$, we have $\gamma(\mathbb{V}) = \gamma(\mathbb{V}_1)\gamma(\mathbb{V}_2)$. Here $\gamma(\mathbb{V}_1) = 1$ since $\mathbb{V}_1 = E(\mathbb{A}_f)$ is coherent, and $\gamma(\mathbb{V}) = \prod_v \gamma(\mathbb{V}_v, q) = (-1)^{\#\Sigma} = -1$ since Σ is assumed to be odd. It follows that $\gamma(\mathbb{V}_2) = -1$. So we get the result for $E_0(s,g,u)$. The formula for $E_a(s,g,u)$ is computed similarly. □

Notation. We introduce the following notations:

$$\begin{aligned} W_a(s,g,u) &= \int_{\mathbb{A}} \delta(wn(b)g)^s \int_{\mathbb{V}_2} r_2(g)\phi_2(x_2,u)\psi(b(uq(x_2)-a))d_u x_2 db, \\ W_{a,v}(s,g,u) &= \int_{F_v} \delta(wn(b)g)^s \int_{\mathbb{V}_{2,v}} r_2(g)\phi_{2,v}(x_2,u)\psi_v(b(uq(x_2)-a))d_u x_2 db, \\ W_{0,v}^\circ(s,g,u) &= \frac{L(s+1,\eta_v)}{L(s,\eta_v)} W_{0,v}(s,g,u). \end{aligned}$$

Here the normalizing factor $\frac{L(s+1,\eta_v)}{L(s,\eta_v)}$ has a zero at $s=0$ when E_v is split, and is equal to π^{-1} at $s=0$ when v is archimedean.

Now we list the values of these local Whittaker functions when $s=0$. They are essentially the Siegel-Weil formula in the coherent case. But in the incoherent case, they will lead to the vanishing of our kernel function at $s=0$.

Proposition 2.6.2. *(1) In the sense of analytic continuation for $s \in \mathbb{C}$,*

$$\begin{aligned} W_0(0,g,u) =& r_2(g)\phi_2(0,u), \\ W_{0,v}^\circ(0,g,u) =& |D_v|^{\frac{1}{2}} |d_v|^{\frac{1}{2}} r_2(g)\phi_{2,v}(0,u). \end{aligned}$$

(2) Assume $a \in F_v^\times$.

(a) If au^{-1} is not represented by $(\mathbb{V}_{2,v}, q_{2,v})$, then $W_{a,v}(0,g,u) = 0$.

(b) Assume that there exists $\xi \in \mathbb{V}_{2,v}$ satisfying $q(\xi) = au^{-1}$. Then

$$W_{a,v}(0,g,u) = \frac{1}{L(1,\eta_v)} \int_{E_v^1} r_2(g)\phi_{2,v}(\xi\tau, u) d\tau.$$

Proof. It is easy to reduce to the case $u = 1$. By Siegel-Weil theorem, we know that all the equalities hold up to constants independent of g and ϕ_v. Hence we only need to check the constants here. Many cases are in the literature (see [KRY] for example). □

Proposition 2.6.3. *$E(0,g,u) = 0$, and thus $I(0,g,\phi) = 0$.*

Proof. It is a direct corollary of Proposition 2.6.2. The Eisenstein series has Fourier expansion

$$E(0,g,u) = \sum_{a \in F} E_a(0,g,u).$$

The vanishing of the constant term easily follows Proposition 2.6.2. For $a \in F^\times$,

$$E_a(0,g,u) = -\prod_v W_{a,v}(0,g,u)$$

is nonzero only if au^{-1} is represented by $(E_v, q_{2,v})$ at each v.

In fact, $\mathbb{B}_v$ is split if and only if there exists a non-zero element $\xi \in \mathbb{V}_{2,v}$ (equivalently, for all nonzero $\xi \in \mathbb{V}_{2,v}$) such that $-q(\xi) \in q(E_v^\times)$. In another word, the following identities hold:

$$\varepsilon(\mathbb{B}_v) = \eta_v(-q(\xi)),$$

where $\varepsilon(\mathbb{B}_v)$ is the Hasse invariant of $\mathbb{B}_v$ and η_v is the quadratic character induced by E_v.

Now the existence of ξ at v such that $q_{2,v}(\xi) = au^{-1}$ is equivalent to

$$\varepsilon(\mathbb{B}_v) = \eta_v(-au^{-1}).$$

If this is true for all v, then

$$\prod_v \varepsilon(\mathbb{B}_v) = \prod_v \eta_v(-au^{-1}) = 1.$$

It contradicts to the incoherence condition that the order of Σ is odd. □

3 Derivatives of kernel functions

In this section, we want to study the derivative of the kernel function for L-series when Σ is odd and π has discrete components of weight 2 at archimedean places. The main content of this section is various local formulae.

In §3.1, we will extend the result in the last section to functions in $\phi \in \mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)_{\mathrm{GO}(F_\infty)}$, the maximal quotient of $\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$ with trivial action by $\mathrm{GO}(F_\infty)$. In §3.2, we decompose the derivative $I'(0,g)$ of the kernel function into a sum of finitely many usual theta series, their derivations, and infinite many local terms $I'(0,g)(v)$ indexed by places of F. Each local term is a period integral of some kernel function $\mathscr{K}^{(v)}(g,(t_1,t_2))$. In §3.3, to simplify the kernel function, we introduce a class of "degenerate" Schwartz functions and proves a non-vanishing result concerning them. We will give some precise formulae for the kernel functions in §3.4 and 3.5. In the last two section we will define and compute the holomorphic projection of $I'(0,g)$.

3.1 Discrete series of weight two at infinite Places

Discrete series of weight two at infinity

If V is positive over an $F\simeq\mathbb{R}$ of even dimension $2d$, the subspace $\mathscr{S}(V,F)^{\mathrm{GO}(F)}$ of functions invariant under $\mathrm{GO}(V)$ will be an representation of $\mathrm{GL}_2(F)$ isomorphic to the discrete series of weight d. In fact this space consists of functions on $V\times F^\times$ of the form:

$$(P_1(|u|q(x))+\mathrm{sgn}(u)P_2(|u|q(x)))e^{-2\pi|u|q(x)}$$

with polynomials P_i on $\mathbb{R}$. The space $\mathscr{S}(V,F)^{\mathrm{GO}(F)}$ can be a quotient of $\mathscr{S}(V\times\mathbb{R}^\times)$ (resp. $\mathscr{S}(V\times\mathbb{R}^\times)^{O(F)}$) via integration over $\mathrm{GO}(F)$:

$$\phi\mapsto\int_{\mathrm{GO}(F)}r(h)\phi dh,\qquad(\text{resp. }\int_{Z(F)}r(h)\phi dh).$$

Thus we have an equality

$$\mathscr{S}(V,F^\times)^{\mathrm{GO}(F)}=\mathscr{S}(V\times F^\times)_{\mathrm{GO}(F)}=\mathscr{S}(V\times F^\times)^{O(F)}_{Z(F)}.$$

where the second (resp. third) the space denote the maximal quotient of $\mathscr{S}(V\times F^\times)$ (resp. $\mathscr{S}(V\times F^\times)^{O(F)}$ with trivial action of $\mathrm{GO}(F)$ (resp. $Z(F)$).

The the *standard Schwartz function* $\phi\in\mathscr{S}(V,\mathbb{R}^\times)$ is the Gaussian

$$\phi^0(x,u)=\frac{1}{2}(1+\mathrm{sgn}(u))e^{-2|u|q(x)}$$

Then one verifies that

$$r(g)\phi(x,u)=W^{(d)}_{uq(x)}(g)$$

where $W_a^{(d)}(g)$ is the standard Whittaker function of weight d for character $e^{2\pi i a x}$:

$$W_a^{(d)}(g) = \begin{cases} |y_0|^{\frac{d}{2}} e^{di\theta} & \text{if } a = 0 \\ |y_0|^{\frac{d}{2}} e^{2\pi i a(x_0+iy_0)} e^{di\theta} & \text{if } ay_0 > 0 \\ 0 & \text{if } ay_0 < 0 \end{cases}$$

for any $a \in \mathbb{R}$ and

$$g = \begin{pmatrix} z_0 & \\ & z_0 \end{pmatrix} \begin{pmatrix} y_0 & x_0 \\ & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$$

in the form of Iwasawa decomposition.

We may extend the result in the last section to functions in

$$\mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)_{\mathrm{GO}(F_\infty)} := \mathscr{S}(\mathbb{V}_\infty, F_\infty^\times)^{\mathrm{GO}(F_\infty)} \otimes \mathscr{S}(\mathbb{V}_f \times \mathbb{A}_f^\times).$$

Theta series

First let us consider the definition of theta series. Let V be a positive definite quadratic space over a totally real field F. Let $\phi \in \mathscr{S}(V(\mathbb{A}) \times \mathbb{A}^\times)_{\mathrm{GO}(F_\infty)}$. There is an open compact subgroup $K \subset \mathrm{GO}(V)(\mathbb{A}_f)$ such that ϕ_f is invariant under the action of K by Weil representation. Denote $\mu_K = F^\times \cap K$. Then μ_K is a subgroup of the unit group $O_F^\times$, and thus is a finitely generated abelian group. Our theta series is of the following form:

$$\theta_\phi(g,h) = \sum_{(x,u)\in\mu_K\backslash(V\times F^\times)} r(g,h)\phi(x,u),$$

The definition here depends on the choice of K. We may normalize this by adding a factor $[O_F^\times : \mu_K]^{-1}$ in front of sum. But we will not use this normalization.

By choosing a different fundamental domain, we can write

$$\theta_\phi(g,h) = \varepsilon_K^{-1} \sum_{u\in\mu_K^2\backslash F^\times} \sum_{x\in V} r(g,h)\phi(x,u).$$

Here $\varepsilon_K = |\{1,-1\} \cap K| \in \{1,2\}$. In particular, $\varepsilon_K = 1$ for K small enough. The summation over u is well-defined since $\phi(x,u) = r(\alpha)\phi(x,u) = \phi(\alpha x, \alpha^{-2}u)$ for any $\alpha \in \mu_K$. It is an automorphic form on $\mathrm{GL}_2(\mathbb{A}) \times \mathrm{GO}(V)(\mathbb{A})$, provided the absolute convergence.

To show the convergence, we claim that the summation over u is actually a finite sum depending on (g,h). For fixed (g,h), there is a compact subset $K' \subset \mathbb{A}_f^\times$ such that $r(g,h)\phi_f(x,u) \neq 0$ only if $u \in K'$. Thus the summation is taken over $u \in \mu_K^2\backslash(F^\times \cap K')$, which is a finite set. In fact, it is a result of finiteness of $\mu_K\backslash(F^\times \cap K')$ and $\mu_K^2\backslash\mu_K$. The former follows from the injection $\mu_K\backslash(F^\times \cap K') \hookrightarrow (\mathbb{A}_f^\times \cap K)\backslash K'$ and compactness of K'. And $\mu_K^2\backslash\mu_K$ is finite because $\mu_K \subset O_F^\times$ is a finitely generated abelian group.

Alternatively, we may construct theta series for above $\phi \in \mathscr{S}(V(\mathbb{A}), \mathbb{A}^\times)$ by some function $\widetilde{\phi} = \phi_\infty \otimes \phi_f \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)^{O(F_\infty)}$ such that

$$\int_{Z(F_\infty)} r(h)\phi dh = \phi^0.$$

In this case,

$$\theta_\phi(g,h) = \int_{Z(F_\infty)/\mu_K} \theta_{\widetilde{\phi}}(g, zh)dz.$$

Kernel functions

For $\phi \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)_{\mathrm{GO}(F_\infty)}$, we may define the mixed Eisenstein-theta series as

$$I(s,g,\phi) = \sum_{\gamma \in P(F)\backslash \mathrm{GL}_2(F)} \delta(\gamma g)^s \sum_{(x_1,u)\in \mu_F \backslash V_1 \times F^\times} r(\gamma g)\phi(x_1,u).$$

If χ has trivial component at infinity, then we may define

$$I(s,g,\phi,\chi) = \mathrm{vol}(K_Z)\int_{T(F)\backslash T(\mathbb{A})/Z(F_\infty)K_Z} \chi(t) I(s,g,r(t,1)\phi)dt.$$

It is clear that $I(s,g,\phi)$ is a finite linear combination of $I(s,g,\phi,\chi)$. The definitions here do not depend on the choice of K. Moreover if

$$\widetilde{\phi} = \widetilde{\phi}_\infty \otimes \phi_f \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)^{O(F_\infty)}$$

such that

$$\phi_\infty = \int_{Z(F_\infty)} r(z)\widetilde{\phi}_\infty dz$$

then

$$I(s,g,\phi) = \int_{Z(F_\infty^\times)/\mu_F} I(s,g,r(z)\widetilde{\phi})dz$$

$$I(s,g,\phi,\chi) = I(s,g,\widetilde{\phi},\chi).$$

Indeed, in the definition of $I(s,g,\phi,\chi)$ in section 2.4, we may decompose the integral over $T(F)\backslash T(\mathbb{A})$ into double integrals over $T(F)\backslash T(\mathbb{A})/Z(F_\infty)K_Z$ and integrals over $Z(F_\infty)K_Z T(F)/T(F)$ to obtain

$$I(s,g,\widetilde{\phi},\chi) = \int_{T(F)\backslash T(\mathbb{A})/Z(F_\infty)K_Z} \chi(t)dt \int_{T(F)\backslash T(F)Z(F_\infty)K_Z} I(s,g,r(tz,1)\widetilde{\phi})dz.$$

The inside integral domain can be identified with

$$T(F)\backslash T(F)Z(F_\infty)K_Z \simeq \mu_Z \backslash Z(F_\infty)K_Z$$

with has a fundamental domain $Z(F_\infty)/\mu_K \times K_Z$. Thus the second integral can be written as

$$\mathrm{vol}(K_Z) \int_{Z(F_\infty)/\mu_K} I(s, g, r(tz, 1)\widetilde{\phi})dz.$$

For this last integral, we write the sum over $V_1 \times F^\times$ in the definition of $I(s, g, \widetilde{\phi})$ as a double sum over $\mu_Z \backslash V_1 \times F^\times$ and a sum of μ_Z. The first sum commutes with integral over $Z(F_\infty)/\mu_K$ while the second the sum collapse with quotient $Z(F_\infty)/\mu_F$ to get an simple integral over $Z(F_\infty)$ which changes $\widetilde{\phi}$ to ϕ.

3.2 Derivatives of Whittaker Functions

We now compute the derivative of the kernel function $I(s, g, \phi)$ for $\phi \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)_{\mathrm{GO}(F_\infty)}$. It suffices to do this for ϕ of the form $\phi = \phi_1 \otimes \phi_2$ with $\phi_i \in \mathscr{S}(\mathbb{V}_i \times \mathbb{A}^\times)_{\mathrm{GO}(V_{i\infty})}$. In this case,

$$I(s, g) = \sum_{u \in \mu_K^2 \backslash F^\times} \theta(g, u, \phi_1) E(s, g, u, \phi_2)$$

where K_Z is an open subset of $Z(\mathbb{A}_f)$ not including such that ϕ_f is invariant under K_Z. It amounts to compute the derivative of the Eisenstein series $E(s, g, u, \phi_2)$. We may further assume that both ϕ_i have standard components $\phi_{i\infty}$ defined in the last subsection. As before we will suppress the dependence on ϕ.

Let us start with the Fourier expansion:

$$E(s, g, u) = \delta(g)^s r_2(g)\phi_2(0, u) - \sum_{a \in F} W_a(s, g, u).$$

Denote by $F(v)$ the set of $a \in F^\times$ that is represented by $(E(\mathbb{A}^v), uq_2^v)$ but not by $(E_v, uq_{2,v})$. Then $F(v) \neq \emptyset$ only if E is non-split at v. By Proposition 2.6.2, $W_{a,v}(0, g, u) = 0$ for any $a \in F(v)$. Then taking the derivative yields

$$W'_a(0, g, u) = W'_{a,v}(0, g, u) W_a^v(0, g, u).$$

It follows that

$$E'(0, g, u) = \log \delta(g) r_2(g)\phi_2(0, u) - W'_0(0, g, u) - \sum_{v \text{ nonsplit}} \sum_{a \in F(v)} W'_{a,v}(0, g, u) W_a^v(0, g, u).$$

Notation. *For any non-split place v, denote the v-part by*

$$E'(0, g, u)(v) := \sum_{a \in F(v)} W'_{a,v}(0, g, u) W_a^v(0, g, u).$$

$$I'(0, g)(v) := \sum_{u \in \mu_K^2 \backslash F^\times} \theta(g, u) E'(0, g, u)(v).$$

Then we have a decomposition

$$I'(0,g) = -\sum_{v \text{ nonsplit}} I'(0,g)(v) - \sum_{u\in\mu_K^2\backslash F^\times} W_0'(0,g,u)\theta(g,u) + \\ + \log\delta(g)\sum_{u\in\mu_K^2\backslash F^\times} r_2(g)\phi_2(0,u)\theta(g,u).$$

We first take care of $I'(0,g)(v)$ for any fixed non-split v. Denote by $B = B(v)$ the nearby quaternion algebra. Then we have a splitting $B = E + Ej$. Let $V = (B,q)$ be the corresponding quadratic space with the reduced norm q, and $V = V_1 + V_2$ be the corresponding orthogonal decomposition. We identify the quadratic spaces $V_{2,w} = \mathbb{V}_{2,w}$ unless $w = v$. It follows that for $a \in F^\times$, $a \in F(v)$ if and only if a is represented by (V_2, uq).

Proposition 3.2.1. *For non-split v,*

$$I'(0,g)(v) = 2 \fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} \mathscr{K}_\phi^{(v)}(g,(t,t))dt,$$

where the integral $\fint$ *employs the Haar measure of total volume one, and*

$$\mathscr{K}_\phi^{(v)}(g,(t_1,t_2)) = \mathscr{K}_{r(t_1,t_2)\phi}^{(v)}(g) = \sum_{u\in\mu_K^2\backslash F^\times}\sum_{y\in V-V_1} k_{r(t_1,t_2)\phi_v}(g,y,u) r(g,(t_1,t_2))\phi^v(y,u).$$

Here for any $y = y_1 + y_2 \in V_v - V_{1v}$,

$$k_{\phi_v}(g,y,u) = \frac{L(1,\eta_v)}{\mathrm{vol}(E_v^1)} r(g)\phi_{1,v}(y_1,u) W'_{uq(y_2),v}(0,g,u).$$

Proof. By Proposition 2.6.2,

$$\begin{aligned} E'(0,g,u)(v) &= \sum_{y_2\in E^1\backslash(V_2-\{0\})} W'_{uq(y_2),v}(0,g,u) W^v_{uq(y_2)}(0,g,u) \\ &= \frac{1}{L^v(1,\eta)} \sum_{y_2\in E^1\backslash(V_2-\{0\})} W'_{uq(y_2),v}(0,g,u) \int_{E^1(\mathbb{A}^v)} r(g)\phi_2^v(y_2\tau,u)d\tau \\ &= \frac{1}{\mathrm{vol}(E_v^1)L^v(1,\eta)} \sum_{y_2\in E^1\backslash(V_2-\{0\})} \int_{E^1(\mathbb{A})} W'_{uq(y_2\tau),v}(0,g,u) r(g)\phi_2^v(y_2\tau,u)d\tau \\ &= \frac{1}{\mathrm{vol}(E_v^1)L^v(1,\eta)} \int_{E^1\backslash E^1(\mathbb{A})} \sum_{y_2\in V_2-\{0\}} W'_{uq(y_2\tau),v}(0,g,u) r(g)\phi_2^v(y_2\tau,u)d\tau. \end{aligned}$$

Therefore, we have the following expression for $I'(0,g)(v)$:

$$I'(0,g)(v) = \frac{1}{\mathrm{vol}(E_v^1)L^v(1,\eta)} \sum_{u\in\mu_K^2\backslash F^\times}\sum_{y_1\in V_1} r(g)\phi_1(y_1,u) \cdot \\ \cdot \int_{E^1\backslash E^1(\mathbb{A})} \sum_{y_2\in V_2-\{0\}} W'_{uq(y_2\tau),v}(0,g,u) r(g)\phi_2^v(y_2\tau,u)d\tau$$

Move two sums inside integral to obtain:

$$\int_{\mathbb{A}^\times E^\times\backslash\mathbb{A}_E^\times}\sum_{u\in\mu_K^2\backslash F^\times}\sum_{\substack{y=y_1+y_2\in V\\ y_2\neq 0}} r(g)\phi_{1,v}(y_1,u)W'_{uq(y_2t^{-1}\bar{t}),v}(0,g,u)r(g)\phi^v(t^{-1}yt,u)dt$$

By definition of k_{ϕ_v} and $\mathscr{K}_\phi^{(v)}$, we have

$$\begin{aligned}I'(0,g)(v) =&\frac{1}{L^v(1,\eta)}\int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\sum_{u\in\mu_K^2\backslash F^\times}\sum_{x\in V-V_1}k_{\phi_v}(g,t^{-1}yt,u)r(g)\phi^v(t^{-1}yt,u)dt\\ =&\frac{1}{L(1,\eta)}\int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\mathscr{K}_\phi^{(v)}(g,(t,t))dt.\end{aligned}$$

Since $\mathrm{vol}(Z(\mathbb{A})T(F)\backslash T(\mathbb{A}))=2L(1,\eta)$, we get the result. Here we have used the relation $k_{\phi_v}(g,t^{-1}yt,u)=k_{r(t,t)\phi_v}(g,y,u)$ in the lemma below. □

Lemma 3.2.2. *The function $k_{\phi_v}(g,y,u)$ behaves like Weil representation under the action of $P(F_v)$ and $E_v^\times\times E_v^\times$. Namely,*

$$\begin{aligned}k_{\phi_v}(m(a)g,y,u) &= |a|^2k_{\phi_v}(g,ay,u),\quad a\in F_v^\times\\ k_{\phi_v}(n(b)g,y,u) &= \psi(buq(y))k_{\phi_v}(g,y,u),\quad b\in F_v\\ k_{\phi_v}(d(c)g,y,u) &= |c|^{-1}k_{\phi_v}(g,y,c^{-1}u),\quad c\in F_v^\times\\ k_{r(t_1,t_2)\phi_v}(g,y,u) &= k_{\phi_v}(g,t_1^{-1}yt_2,q(t_1t_2^{-1})u),\quad (t_1,t_2)\in E_v^\times\times E_v^\times\end{aligned}$$

Proof. These identities follow from the definition of Weil representation and some transformation of integrals. We will only verify the first identity. By definition, we can compute the transformation of $m(a)$ on Whittaker function directly:

$$\begin{aligned}W_{a_0,v}(s,m(a)g,u) &= \int_{F_v}\delta(wn(b)m(a)g)^s r_2(wn(b)m(a)g)\phi_{2,v}(0,u)\psi_v(-a_0b)db\\ &= \int_{F_v}\delta(m(a^{-1})wn(ba^{-2})g)^s r_2(m(a^{-1})wn(ba^{-2})g)\phi_{2,v}(0,u)\psi_v(-a_0b)db\\ &= |a|^{-s}\int_{F_v}\delta(wn(ba^{-2})g)^s r_2(wn(ba^{-2})g)\phi_{2,v}(0,u)|a|^{-1}\eta_v(a)\psi_v(-a_0b)db\\ &= |a|^{-s-1}\eta_v(a)\int_{F_v}\delta(wn(b)g)^s r_2(wn(b)g)\phi_{2,v}(0,u)\psi_v(-a_0a^2b)|a|^2db\\ &= |a|^{-s+1}\eta_v(a)W_{a^2a_0,v}(s,g,u).\end{aligned}$$

It follows that

$$W'_{a_0,v}(0,m(a)g,u)=|a|\eta_v(a)W'_{a^2a_0,v}(0,g,u).$$

This implies the result by combining

$$r(m(a)g)\phi_{1,v}(y_1,u)=|a|\eta_v(a)\phi_{1,v}(ay_1,u).$$

□

Now we rewrite the second and the third summations, which is less complicated than the first one.

Proposition 3.2.3.

$$I'(0,g) = -\sum_{v \text{ nonsplit}} I'(0,g)(v) + 2\log\delta(g)\sum_{u\in\mu_K^2\backslash F^\times}\sum_{y\in V_1} r(g)\phi(y,u)$$
$$- c_0\sum_{u\in\mu_K^2\backslash F^\times}\sum_{y\in V_1} r(g)\phi(y,u) - \sum_v\sum_{u\in\mu_K^2\backslash F^\times}\sum_{y\in V_1} c_{\phi_v}(g,y,u)r(g^v)\phi^v(y,u),$$

where

$$c_0 = \frac{d}{ds}|_{s=0}\left(\log\frac{L(s,\eta)}{L(s+1,\eta)}\right)$$

is a constant, and

$$c_{\phi_v}(g,y,u) = r_1(g)\phi_{1,v}(y,u)\frac{W_{0,v}^{\circ}{}'(0,g,u)}{|D_v|^{\frac{1}{2}}|d_v|^{\frac{1}{2}}} + \log\delta(g_v)r(g)\phi_v(y,u).$$

Moreover, $c_{\phi_v} = 0$ identically for all archimedean places v and almost all non-archimedean places v.

Proof. Take derivative on

$$W_0(s,g,u) = \frac{L(s,\eta)}{L(s+1,\eta)}W_0^\circ(s,g,u) = \frac{L(s,\eta)}{L(s+1,\eta)}\prod_v W_{0,v}^\circ(s,g,u).$$

By Proposition 2.6.2,

$$W_0'(0,g,u) = \frac{d}{ds}|_{s=0}\left(\frac{L(s,\eta)}{L(s+1,\eta)}\right)W_0^\circ(0,g,u) + \frac{L(0,\eta)}{L(1,\eta)}\sum_v W_{0,v}^\circ{}'(0,g,u)\prod_{v'\neq v}W_{0,v'}^\circ(0,g,u)$$
$$= \frac{d}{ds}|_{s=0}\left(\log\frac{L(s,\eta)}{L(s+1,\eta)}\right)r(g)\phi(0,u) + \sum_v\frac{W_{0,v}^\circ{}'(0,g,u)}{|D_v|^{\frac{1}{2}}|d_v|^{\frac{1}{2}}}r(g^v)\phi^v(0,u).$$

It gives the desired equality. The map

$$\delta(g)^s r(g)\phi_2(0,u) \mapsto W_0(s,g,u,\phi_2), \quad I(s,\eta)\to I(-s,\eta)$$

is an intertwining operator. For archimedean places v and almost all non-archimedean places v, one has the local component

$$W_{0,v}^\circ(s,g,u,\phi_{2,v}) = \delta(g)^{-s}r(g)\phi_2(0,u).$$

And thus we have $c_{\phi_v} = 0$. See [KRY]. □

3.3 Degenerate Schwartz functions at non-archimedean places

In this subsection we introduce a class of "degenerate" Schwartz functions at a non-archimedean place. It is generally very difficult to obtain an explicit formula of $I'(0,g)(v)$ for a ramified finite prime v. When we choose a degenerate Schwartz function at v, the function $I'(0,g)(v)$ turns out to be easier to control as we will see in the next subsection.

Define

$$\mathscr{S}^0(\mathbb{B}_v \times F_v^\times) = \begin{cases} \{\phi_v \in \mathscr{S}(\mathbb{B}_v \times F_v^\times) : \phi_v|_{\mathbb{B}_v{}^{\mathrm{sing}} \cup E_v \times F_v^\times}) = 0\}, & \text{if } E_v/F_v \text{ is nonsplit;} \\ \{\phi_v \in \mathscr{S}(\mathbb{B}_v \times F_v^\times) : \phi_v|_{\mathbb{B}_v{}^{\mathrm{sing}} \times F_v^\times} = 0\}, & \text{if } E_v/F_v \text{ is split.} \end{cases}$$

Here $\mathbb{B}_v{}^{\mathrm{sing}} = \{x \in \mathbb{B}_v : q(x) = 0\}$.

By compactness, for any $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$, there exists a constant $C > 0$ such that $\phi_v(x, F_v^\times) = 0$ if $|v(q(x))| > C$. The following result will be useful when we extend our final results to general ϕ_v.

Proposition 3.3.1. *Let v be a non-archimedean place. Then for any nonzero $\mathrm{GL}_2(F_v)$-equivariant homomorphism $\mathscr{S}(\mathbb{B}_v \times F_v^\times) \to \pi_v$, the image of $\mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ in π_v is nonzero.*

We will prove a more general result. For simplicity, let F be a non-archimedean local field and let (V, q) be a non-degenerate quadratic space over F of even dimension. Then we have the Weil representation of $\mathrm{GL}_2(F)$ on $\mathscr{S}(V \times F^\times)$, the space of Bruhat-Schwartz functions on $V \times F^\times$. Let $\alpha : \mathscr{S}(V \times F^\times) \longrightarrow \sigma$ be a surjective morphism to an irreducible and admissible representation of $\mathrm{GL}_2(F)$. We will prove the following result which obviously implies Proposition 3.3.1.

Proposition 3.3.2. *Let W be a proper subspace of V of even dimension. Assume that σ is not one dimensional, and that in case $W \neq 0$, W is non-degenerate and its orthogonal complement W' is anisotropic. Then there is a function $\phi \in \mathscr{S}(V \times F^\times)$ such that $\alpha(\phi) \neq 0$ and that the support $\mathrm{supp}(\phi)$ of ϕ contains only elements (x, u) such that $q(x) \neq 0$ and*

$$W(x) := W + Fx$$

is non-degenerate of dimension $\dim W + 1$.

Let us start with the following Proposition which allows us to modify any test function to a function with support at points $(x, u) \in V \times F^\times$ with components x of nonzero norm $q(x) \neq 0$.

Proposition 3.3.3. *Let $\phi \in \mathscr{S}(V \times F^\times)$ be an element with nonzero image in σ. Then there is a function $\widetilde{\phi} \in \mathscr{S}(V \times F^\times)$ supported on*

$$\mathrm{supp}(\widetilde{\phi}) \subset \mathrm{supp}(\phi) \cap \left(V_{q \neq 0} \times F^\times\right).$$

The key to prove this proposition is the following lemma. It is well-known but we give a proof for readers' convenience.

Lemma 3.3.4. *Let π be an irreducible and admissible representation of* $\mathrm{GL}_2(F)$ *with* $\dim > 1$*. Then the group $N(F)$ of matrixes $n(t) := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ does not have any non-zero invariant on π.*

Proof. If there exists such an invariant vector $v \neq 0$, then by the smoothness, v is also fixed by some compact open subgroup. In particular, v is fixed by some element $\gamma \in SL_2(F) - B(F)$. Then v is in fact SL_2-invariant since $SL_2(F)$ is generated by N and any one element γ in $SL_2(F) - B(F)$. Then such π must be one dimensional. Contradiction! □

Proof of Proposition 3.3.3. Applying the lemma above, we obtain elements $t \in F$ such that

$$\sigma(n(t))\alpha(\phi) - \alpha(\phi) \neq 0.$$

The left hand side is equal to $\alpha(\widetilde{\phi})$ with

$$\widetilde{\phi} = n(t)\phi - \phi.$$

By definition, we have

$$\widetilde{\phi}(x, u) = (\psi(tuq(x)) - 1)\phi(x).$$

Thus such a $\widetilde{\phi}$ has support

$$\mathrm{supp}(\widetilde{\phi}) \subset \mathrm{supp}(\phi) \cap \left(V_{q\neq 0} \times F^\times\right).$$

□

Proof of Proposition 3.3.2. Choose any ϕ such that $\alpha(\phi) \neq 0$. If $W = 0$, then we apply Proposition 3.3.3 to obtain a function $\widetilde{\phi}$ satisfying the proposition. Thus we assume that $W \neq 0$. Since W' is anisotropic, we have an orthogonal decomposition $V = W \oplus W'$, and an identification $\mathscr{S}(V \times F^\times) = \mathscr{S}(W \times F^\times) \otimes_{\mathscr{S}(F^\times)} \mathscr{S}(W' \times F^\times)$. The action of $\mathrm{GL}_2(F)$ is given by actions on $\mathscr{S}(W \times F^\times)$ and $\mathscr{S}(W' \times F^\times)$ respectively. We may assume that ϕ is a pure tensor:

$$\phi = f \otimes f', \qquad f \in \mathscr{S}(W \times F^\times), \quad f' \in \mathscr{S}(W' \times F^\times).$$

Since W' is anisotropic, $\mathscr{S}(W'_{q\neq 0} \times F^\times)$ is a subspace in $\mathscr{S}(W' \times F^\times)$ with quotient $\mathscr{S}(F^\times)$. The quotient map is given by evaluation at $(0, u)$. Thus

$$\mathscr{S}(W' \times F^\times) = \mathscr{S}(W'_{q\neq 0} \times F^\times) + w\mathscr{S}(W'_{q\neq 0} \times F^\times)$$

as w acts as the Fourier transform up to a scale multiple. In this way, we may write

$$f' = f'_1 + wf'_2, \qquad f'_i \in \mathscr{S}(W'_{q\neq 0} \times F^\times).$$

Then we have decomposition

$$\phi = \phi_1 + w\phi_2. \qquad \phi_1 := f \otimes f'_1, \phi_2 = w^{-1} f \otimes f'_2.$$

One of $\alpha(\phi_i) \neq 0$. Thus we may replace ϕ by this ϕ_i to conclude that the support of ϕ consists of points $x = (w, w')$ with $W(x) = W \oplus Fw'$ which is non-degenerate. Applying Proposition 3.3.3 again, we may further assume that ϕ has support on $V_{q\neq 0} \times F^\times$. □

3.4 Non-archimedean components

Assume that v is a non-archimedean place non-split in E. Resume the notations in the last section. We now compute the local kernel function:

$$k_{\phi_v}(g,y,u)=\frac{L(1,\eta_v)}{\mathrm{vol}(E_v^1)}r(g)\phi_{1,v}(y_1,u)W'_{uq(y_2),v}(0,g,u),\quad y=y_1+y_2\in V_v-V_{1v}.$$

Recall that $\mathrm{vol}(O_{F_v})=|d_v|_v^{\frac{1}{2}}$ and $\mathrm{vol}(O_{E_v})=|D_v|^{\frac{1}{2}}|d_v|_v$ under the self-dual measures. Here $d_v\in O_{F_v}$ is the local different of F_v, and $D_v\in O_{F_v}$ is the local discriminant of E_v/F_v.

Proposition 3.4.1. *Assume $a\in F(v)$ for a finite prime v.*

(1) If $v\nmid 2$ or E is unramified at v, then for $g=1$,

$$W'_{a,v}(0,1,u)\ =\ |d_v|^{\frac{1}{2}}\log N_v\sum_{n=0}^{v(ad_v)}N_v^n\int_{D_n}\phi_{2,v}(x_2,u)d_ux_2,$$

and thus

$$k_{\phi_v}(1,y,u)=\frac{L(1,\eta_v)}{\mathrm{vol}(E_v^1)}|d_v|^{\frac{1}{2}}\log N_v\ \phi_{1,v}(y_1,u)\sum_{n=0}^{v(uq(y_2)d_v)}N_v^n\int_{D_n}\phi_{2,v}(x_2,u)d_ux_2.$$

Here the domain

$$D_n=\{x_2\in\mathbb{V}_{2,v}:uq(x_2)\in p_v^nd_v^{-1}\}$$

is an O_{E_v}-lattice of $\mathbb{V}_{2,v}$, and the measure d_ux_2 is the self-dual measure of $(\mathbb{V}_{2,v},uq)$ which gives $\mathrm{vol}(O_{E_v}x_2)=|D_v|^{\frac{1}{2}}|d_vuq(x_2)|$ for any $x_2\in\mathbb{V}_{2,v}$.

(2) In all cases including $v\mid 2$, if $\phi_v\in\mathscr{S}^0(\mathbb{V}_v\times F_v^\times)$, then $k_{\phi_v}(1,y,u)$ extends to a Schwartz function of $(y,u)\in V_v\times F_v^\times$.

Proof. It suffices to verify the formulae for Whittaker functions under the condition that $u=1$. The general case is obtained by replacing q by uq. We will drop the index u to simplify the notation. Recall

$$W_{a,v}(s,1)=\int_{F_v}\delta(wn(b))^s\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db.$$

By

$$\delta(wn(b))=\begin{cases}1 & \text{if } b\in O_{F_v},\\ |b|^{-1} & \text{otherwise,}\end{cases}$$

we will split the integral over F_v into the sum of an integral over O_{F_v} and an integral over $F_v-O_{F_v}$. Then

$$\begin{aligned}W_{a,v}(s,1)=&\int_{O_{F_v}}\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\&+\int_{F_v-O_{F_v}}|b|^{-s}\int_{\mathbb{V}_{2,v}}\phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\end{aligned}$$

The second integral can be decomposed as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\int_{p_v^{-n}-p_v^{-n+1}} N_v^{-ns}\int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\
=&\sum_{n=1}^{\infty}\int_{p_v^{-n}} N_v^{-ns}\int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\
&-\sum_{n=1}^{\infty}\int_{p_v^{-(n-1)}} N_v^{-ns}\int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db.
\end{aligned}$$

Combine with the first integral to obtain

$$\begin{aligned}
W_{a,v}(s,1)=&\sum_{n=0}^{\infty}\int_{p_v^{-n}} N_v^{-ns}\int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\
&-\sum_{n=0}^{\infty}\int_{p_v^{-n}} N_v^{-(n+1)s}\int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db\\
=&(1-N_v^{-s})\sum_{n=0}^{\infty}N_v^{-ns}\int_{p_v^{-n}}\int_{\mathbb{V}_{2,v}} \phi_{2,v}(x_2)\psi_v(b(q(x_2)-a))dx_2db.
\end{aligned}$$

As for the last double integral, change the order of the integration. The integral on b is nonzero if and only if $q(x_2)-a\in p_v^n d_v^{-1}$. Here d_v is the local different of F over $\mathbb{Q}$, and also the conductor of ψ_v. Denote

$$D_n(a)=\{x_2\in\mathbb{V}_{2,v}: q(x_2)-a\in p_v^n d_v^{-1}\}.$$

Then we have

$$\begin{aligned}
W_{a,v}(s,1) &= (1-N_v^{-s})\sum_{n=0}^{\infty}N_v^{-ns}\mathrm{vol}(p_v^{-n})\int_{D_n(a)}\phi_{2,v}(x_2)dx_2\\
&= |d_v|^{\frac12}(1-N_v^{-s})\sum_{n=0}^{\infty}N_v^{-ns+n}\int_{D_n(a)}\phi_{2,v}(x_2)dx_2.
\end{aligned}$$

We will see that $D_n(a)$ is empty for n large enough by the condition that $a\in F(v)$. Then the sum above have only finitely many terms, and thus $W_{a,v}(0,1)=0$ just by the vanishing of the factor $1-N_v^{-s}$. Hence,

$$W'_{a,v}(0,1) = |d_v|^{\frac12}\log N_v\sum_{n=0}^{\infty}N_v^{n}\int_{D_n(a)}\phi_{2,v}(x_2)dx_2.$$

We first show (1). We claim that $v(q(x_2)-a)=\min\{v(a),v(q(x_2))\}$ for any $x_2\in\mathbb{V}_{2,v}$. Actually, if this is not true, we will have $v(a^{-1}q(x_2))=0$ and $v(1-a^{-1}q(x_2))>0$. Then

$a^{-1}q(x_2) \in 1 + p_v \subset q(E_v^\times)$, and thus a is represented by $(\mathbb{V}_{2,v}, q)$, which contradicts to the assumption that $a \in F(v)$. This simple fact implies that

$$D_n(a) = \{x_2 \in \mathbb{V}_{2,v} : q(x_2) \in p_v^n d_v^{-1}, a \in p_v^n d_v^{-1}\}.$$

We see that $D_n(a)$ is empty if $v(a) < n - v(d_v)$. Otherwise, $D_n(a) = D_n$. It proves (1).

Now we consider (2). If v satisfies the assumption of (1), then by the results we know that $W'_{a,v}(0,1) = 0$ if $v(a)$ is too small. On ther other hand, we have $\phi_{2,v}(D_n, u) = 0$ if n is large enough by the vanishing property of the Schwartz function. It follows that $W'_{a,v}(0,1)$ is constant when $v(a)$ is large enough. By both facts, we can obtain the result.

If v divides 2 and ramifies in E, then $D_n(a)$ has no simple expression as above since we don't have $v(q(x_2) - a) = \min\{v(a), v(q(x_2))\}$ any more. However, we have

$$v(q(x_2) - a) \leq \min\{v(a), v(q(x_2))\} + v(8)$$

by a similar argument: if this were not true, then $v(a^{-1}q(x_2)) = 0$ and $v(1 - a^{-1}q(x_2)) > v(8)$. It follows that $a^{-1}q(x_2) \in 1 + 8p_v \subset q(E_v^\times)$, and thus a is represented by (E_v, q), which contradicts the assumption that $a \in F(v)$.

Therefore, $D_n(a)$ is non-empty only if $v(a) > -v(8)$, and

$$D_n(a) \subset \{x_2 \in \mathbb{V}_{2,v} : 8q(x_2) \in p_v^n d_v^{-1}\}.$$

It is easy to obtain (2) by a similar argument.

□

Corollary 3.4.2. *Assume that v is a finite prime nonsplit in E such that:*

- *$\mathbb{B}_v$ is split at v;*
- *$v \nmid 2$ if E_v/F_v is ramified;*
- *the local different d_v is trivial (which makes the additive character ψ_v unramified);*
- *ϕ_v is the characteristic function of $O_{\mathbb{B}_v} \times O_{F_v}^\times$.*

Then $k_{\phi_v}(1, y, u) \neq 0$ only if $(y, u) \in O_{B_v} \times O_{F_v}^\times$. Here B_v the quaternion over F_v that is non-isomorphic to $\mathbb{B}_v$. Under this condition, we have

$$k_{\phi_v}(1, y, u) = \frac{v(q(y_2)) + 1}{2} \log N_v.$$

Proof. Note that $\phi_{1,v}(y_1, u)$ is a factor of $k_{\phi_v}(1, y, u)$. To make it nonzero, we need to assume $(y_1, u) \in O_{E_v} \times O_{F_v}^\times$. Then

$$k_{\phi_v}(1, y, u) = \frac{L(1, \eta_v)}{\mathrm{vol}(E_v^1)} \log N_v \sum_{n=0}^{v(q(y_2))} N_v^n \int_{D_n} \phi_{2,v}(x_2, u) d_q x_2 = \frac{L(1, \eta_v)}{\mathrm{vol}(E_v^1)} \log N_v \sum_{n=0}^{v(q(y_2))} N_v^n \mathrm{vol}(D_n)$$

with

$$D_n = \{x_2 \in \mathbb{V}_{2,v} : q(x_2) \in p_v^n\}.$$

It is nonzero only if $v(q(y_2)) \geq 0$, which we now assume.

Recall $\mathbb{V}_{2,v} = E_v \mathfrak{j}_v$. Since $\mathbb{B}_v$ is a matrix algebra, we can assume that $q(\mathfrak{j}_v) \in O_{F_v}^\times$. Then

$$D_n = \{x_2 \mathfrak{j}_v : x_2 \in E_v, q(x_2) \in p_v^n\}.$$

If E is unramified at v, we have $D_n = p_v^{[\frac{n+1}{2}]} O_{E_v} \mathfrak{j}_v$ and $\mathrm{vol}(D_n) = N_v^{-2[\frac{n+1}{2}]}$. Then $N_v^n \mathrm{vol}(D_n) = 1$ for even n, and $N_v^n \mathrm{vol}(D_n) = N_v^{-1}$ for odd n. Note that $v(q(y_2))$ is odd since B_v is nonsplit. Hence,

$$k_{\phi_v}(1, y, u) = L(1, \eta_v) \log N_v \frac{v(q(y_2)) + 1}{2}(1 + N_v^{-1}) = \frac{v(q(y_2)) + 1}{2} \log N_v.$$

If E is ramified at v, then $D_n = p_v^{\frac{n}{2}} O_{E_v} \mathfrak{j}_v$ and $\mathrm{vol}(D_n) = N_v^{-n} |D_v|^{\frac{1}{2}}$. By $\mathrm{vol}(E_v^1) = 2|D_v|^{\frac{1}{2}}$, we still get the result.

□

3.5 Archimedean Places

For an archimedean place v, the quaternion algebra $\mathbb{B}_v$ is isomorphic to the Hamiltonian quaternion. We will compute $k_{\phi_v}(g, y, u)$ for standard ϕ_v of weight 2 introduced in Section 3.1. The computation here is done by [KRY].

The result involves the exponential integral Ei defined by

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \frac{e^t}{t} dt, \quad z \in \mathbb{C}.$$

Another expression is

$$\mathrm{Ei}(z) = \gamma + \log(-z) + \int_0^z \frac{e^t - 1}{t} dt,$$

where γ is the Euler constant. It follows that it has a logarithmic singularity near 0. This fact is useful when we compare the result here with the archimedean local height, since we know that Green's functions have a logarithmic singularity.

Proposition 3.5.1.

$$k_{\phi_v}(g, y, u) = \begin{cases} -\frac{1}{2}\mathrm{Ei}(4\pi u q(y_2) y_0) \ |y_0| e^{2\pi i u q(y)(x_0 + i y_0)} e^{2i\theta} & \text{if } \ u y_0 > 0 \\ 0 & \text{if } \ u y_0 < 0 \end{cases}$$

for any

$$g = \begin{pmatrix} z_0 & \\ & z_0 \end{pmatrix} \begin{pmatrix} y_0 & x_0 \\ & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{GL}_2(F_v)$$

in the form of the Iwasawa decomposition.

Proof. It suffices to show the formula in the case $g = 1$. The general case is obtained by Proposition 3.2.2 and the fact that $r(k_\theta)\phi_v = e^{2i\theta}\phi_v$.

Now we show that

$$k_{\phi_v}(1, y, u) = \begin{cases} -\frac{1}{2}\mathrm{Ei}(4\pi u q(y_2))e^{-2\pi u q(y)} & \text{if } u > 0; \\ 0 & \text{if } u < 0. \end{cases}$$

It amounts to show that, for any $a \in F(v)$,

$$W'_{a,v}(0, 1, u) = \begin{cases} -\pi e^{-2\pi a}\mathrm{Ei}(4\pi a) & \text{if } u > 0; \\ 0 & \text{if } u < 0. \end{cases}$$

Assume $u > 0$ since the case $u < 0$ is trivial. Then $a < 0$ by the condition $a \in F(v)$. We compute explicitly as follows. First of all, it is easy to check $\delta(wn(b)) = 1/(1 + b^2)$. It follows that

$$\begin{aligned} W_{a,v}(s, 1, u) &= \int_{F_v} \delta(wn(b))^s \int_{V_{2v}} \phi_{2,v}(x_2, u)\psi_v(b(uq(x_2) - a))d_u x_2 db \\ &= \int_{\mathbb{R}} \left(\frac{1}{1+b^2}\right)^{\frac{s}{2}} \int_{\mathbb{C}} e^{-2\pi u q(x_2) + 2\pi i b(uq(x_2) - a)} d_u x_2 db \end{aligned}$$

Take an isometry of quadratic space $(V_{2v}, uq) \simeq (\mathbb{C}, |\cdot|^2)$ to obtain

$$\begin{aligned} &\int_{\mathbb{R}} \left(\frac{1}{1+b^2}\right)^{\frac{s}{2}} \int_{\mathbb{C}} e^{-2\pi|x_2|^2} e^{2\pi i b(|x_2|^2 - a)} dx_2 db \\ =& \int_{\mathbb{R}} \left(\frac{1}{1+b^2}\right)^{\frac{s}{2}} \int_{\mathbb{C}} e^{-2\pi(1-ib)|x_2|^2} e^{-2\pi i a b} dx_2 db = \int_{\mathbb{R}} \left(\frac{1}{1+b^2}\right)^{\frac{s}{2}} \frac{1}{1-ib} e^{-2\pi i a b} db \\ =& \int_{\mathbb{R}} (1+ib)^{-\frac{s}{2}}(1-ib)^{-\frac{s}{2}-1} e^{-2\pi i a b} db. \end{aligned}$$

By the computation in [KRY], page 19,

$$\frac{d}{ds}|_{s=0} \int_{\mathbb{R}} (1+ib)^{-\frac{s}{2}}(1-ib)^{-\frac{s}{2}-1} e^{-2\pi i a b} db = -\pi e^{-2\pi a} Ei(4\pi a).$$

Thus

$$W'_{a,v}(0, 1, u) = -\pi e^{-2\pi a}\mathrm{Ei}(4\pi a).$$

□

3.6 Holomorphic projection

In this subsection we consider the general theory of holomorphic projection which we will apply to the form $I'(0, g, \chi)$ in the next subsection. Denote by $\mathscr{A}(\mathrm{GL}_2(\mathbb{A}), \omega)$ the space of automorphic forms of central character ω, by $\mathscr{A}_0(\mathrm{GL}_2(\mathbb{A}),\omega)$ the subspace of cusp forms, and by $\mathscr{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}),\omega)$ the subspace of holomorphic cusp forms of parallel weight two.

The usual Petersson inner product is just

$$(f_1, f_2) = \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} f_1(g)\overline{f_2(g)}dg, \quad f_1, f_2 \in \mathscr{A}(\mathrm{GL}_2(\mathbb{A}),\omega).$$

Denote by $\mathscr{P}r' : \mathscr{A}(\mathrm{GL}_2(\mathbb{A}),\omega) \to \mathscr{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}),\omega)$ the orthogonal projection. Namely, for any $f \in \mathscr{A}(\mathrm{GL}_2(\mathbb{A}),\omega)$, the image $\mathscr{P}r'(f)$ is the unique form in $\mathscr{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}),\omega)$ such that

$$(\mathscr{P}r'(f), \varphi) = (f, \varphi), \quad \forall \varphi \in \mathscr{A}_0^{(2)}(\mathrm{GL}_2(\mathbb{A}),\omega).$$

We simply call $\mathscr{P}r'(f)$ the holomorphic projection of f. Apparently $\mathscr{P}r'(f) = 0$ if f is an Eisenstein series.

For any automorphic form f for $\mathrm{GL}_2(\mathbb{A})$ we define a Whittaker function

$$f_{\psi,s}(g) = (4\pi)^{\deg F} W^{(2)}(g_\infty) \int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \delta(g)^s f_\psi(g_f h)\overline{W^{(2)}(h)}dh.$$

Here $W^{(2)}$ is the standard holomorphic Whittaker function of weight two at infinity, and f_ψ denotes the Whittaker function of f. As $s \to 0$, the limit of the integral is holomorphic at $s = 0$.

Proposition 3.6.1. *Let $f \in \mathscr{A}(\mathrm{GL}_2(\mathbb{A}),\omega)$ be a form with asymptotic behavior*

$$f\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) = O(|a|^{1-\epsilon})$$

as $|a| \to \infty$ for some $\epsilon > 0$. Then the holomorphic projection $\mathscr{P}r'(f)$ has Whittaker function

$$\mathscr{P}r'(f)_\psi(g_f g_\infty) = \lim_{s\to 0} f_{\psi,s}(g).$$

Proof. For any Whittaker function W of $\mathrm{GL}_2(\mathbb{A})$ with decomposition $W(g) = W^{(2)}(g_\infty)W_f(g_f)$ such that $W^{(2)}(g_\infty)$ is standard holomorphic of weight 2 and that $W_f(g_f)$ compactly supported modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the Poincaré series is defined as follows:

$$\varphi_W(g) := \lim_{s\to 0+} \sum_{\gamma \in Z(F)N(F)\backslash G(F)} W(\gamma g)\delta(\gamma g)^s,$$

where

$$\delta(g) = |a_\infty/d_\infty|^{\frac{1}{2}}, \qquad g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, \qquad k \in U$$

where U is the standard maximal compact subgroup of $\mathrm{GL}_2(\mathbb{A})$. Assume that W and f have the same central character. Since f has asymptotic behavior as in the proposition, their inner product can be computed as follows:

$$\begin{aligned}(f,\varphi_W) &= \int_{Z(\mathbb{A})\mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g)\bar{\varphi}_W(g)dg\\ &= \lim_{s\longrightarrow 0}\int_{Z(\mathbb{A})N(F)\backslash \mathrm{GL}_2(\mathbb{A})} f(g)\bar{W}(g)\delta(g)^s dg\\ &= \lim_{s\longrightarrow 0}\int_{Z(\mathbb{A})N(\mathbb{A})\backslash \mathrm{GL}_2(\mathbb{A})} f_\psi(g)\bar{W}(g)\delta(g)^s dg. \end{aligned} \tag{3.6.1}$$

We may apply this formula to $\mathscr{P}r'(f)$ which has the same inner product with φ_W as f. Write

$$\mathscr{P}r'(f)_\psi(g) = W^{(2)}(g_\infty)\mathscr{P}r'(f)_\psi(g_f).$$

Then the above integral is a product of integrals over finite places and integrals at infinite places:

$$\int_{Z(\mathbb{R})N(\mathbb{R})\backslash \mathrm{GL}_2(\mathbb{R})} |W^{(2)}(g)|^2 dg = \int_0^\infty y^2 e^{-4\pi y} dy/y^2 = (4\pi)^{-1}.$$

In other words, we have

$$(f,\varphi_W) = (4\pi)^{-g}\int_{Z(\mathbb{A}_f)N(\mathbb{A}_f)\backslash \mathrm{GL}_2(\mathbb{A}_f)} \mathscr{P}r'(f)_\psi(g_f)\bar{W}(g_f)dg_f. \tag{3.6.2}$$

As $\bar{W}$ can be any Whittaker function with compact support modulo $Z(\mathbb{A}_f)N(\mathbb{A}_f)$, the combination of above formulae give the proposition. □

We introduce an operator $\mathscr{P}r$ formally defined on the function space of $N(F)\backslash \mathrm{GL}_2(\mathbb{A})$. For any function $f : N(F)\backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$, denote as above

$$f_{\psi,s}(g) = (4\pi)^{\deg F} W^{(2)}(g_\infty)\int_{Z(F_\infty)N(F_\infty)\backslash \mathrm{GL}_2(F_\infty)} \delta(g)^s f_\psi(g_f h)\overline{W^{(2)}(h)}dh$$

if it has meromorphic continuation around $s = 0$. Here f_ψ denotes the first Fourier coefficient of f. Denote

$$\mathscr{P}r(f)_\psi(g_f g_\infty) = \widetilde{\lim}_{s\to 0} f_{\psi,s}(g),$$

where $\widetilde{\lim}_{s\to 0}$ denotes the constant term of the Laurent expansion at $s = 0$. Finally, we write

$$\mathscr{P}r(f)(g) = \sum_{a\in F^\times} \mathscr{P}r(f)_\psi(d^*(a)g).$$

The the above result is just $\mathscr{P}r(f) = \mathscr{P}r'(f)$. In general, $\mathscr{P}r(f)$ is not automorphic when f is automorphic but fails the growth condition of Proposition 3.6.1.

Now we want to consider the holomorphic projection of $I'(0,g,\phi)$ for ϕ with standard infinite component at infinity. First, let us compute the asymptotic behavior of $I(s,g,\chi)$ near a cusp for small s. This function is a finite linear combination of $I(s,g,\chi)$'s:

$$I(s,g,\chi) = \int_{T(F)Z(F_\infty)\backslash T(\mathbb{A})} I(s,g,r(t,1)\phi)\chi(t)dt$$

with central character $\chi|_{\mathbb{A}_F^\times}$. Thus we can define and compute the holomorphic projection using Proposition 3.6.1.

Proposition 3.6.2. *If χ does not factor through norm $\mathrm{N}_{E/F}: \mathbb{A}_E^\times \longrightarrow \mathbb{A}^\times$, then $I(s,g,\chi)$ has an asymptotic $O(|a|^{1-\epsilon})$. Otherwise, let χ be induced from two different characters μ_i on $F^\times\backslash\mathbb{A}^\times$:*

$$\chi = \mu_1 \circ \mathrm{N}_{E/F} = \mu_2 \circ \mathrm{N}_{E/F}.$$

Then there is an Eisenstein series $J(s,g,\chi)$ for the characters

$$\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto |a/b|^{1/2\pm s}\mu_i^{-1}(ab)$$

such that $I(s,g,\chi) - J(s,g,\chi)$ has asymptotic behavior $O(|a|^{1-\epsilon})$.

Proof. Recall that $I(s,g,\chi)$ can be rewritten as an integration of $I(s,g)$:

$$I(s,g,\chi) = \int_{T(F)Z(F_\infty)\backslash T(\mathbb{A})} I(s,g,r(t,1)\phi)\chi(t)dt$$

and that

$$I(s,g) = \sum_{u\in\mu_K^2\backslash F^\times} \theta(g,u)E(s,g,u).$$

It suffices to get an asymptotic behavior for the theta series and the Eisenstein series for left action by diagonal elements $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. In their explicit Fourier expansions, the non-constant term exponentially decays in as $|a/b| \to \infty$. The constant terms have asymptotical behavior:

$$\theta_0\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} g, u\right) = |a/b|^{1/2}\theta_0(g, a^{-1}b^{-1}u) = r(g)\phi(0, a^{-1}b^{-1}u)|a/b|^{1/2}$$

$$E_0\left(s, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} g, u\right) = |a/b|^{1/2+s}f(g, a^{-1}b^{-1}u) + |a/b|^{1/2-s}\widetilde{f}(g, a^{-1}b^{-1}u)$$

where

$$f(g,u) = \delta(g)^s r_2(g)\phi_2(0,u), \qquad \widetilde{f}(g,u) = -\int_{N(\mathbb{A})} \delta(wmg)^s r_2(wng)\phi_2(0,u).$$

It follows that $I(s,g)$ has the following asymptotic behavior modulo a function with exponential decay,

$$I_0\left(s, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} g\right) = |a/b|^{1+s} \sum_{\mu_K^2\backslash F^\times} \theta_0 f(g, a^{-1}b^{-1}u) + |a/b|^{1-s} \sum_{\mu_K^2\backslash F^\times} \theta_0 \widetilde{f}(g, a^{-1}b^{-1}u).$$

Similarly $I(s,g,\chi)$ has an asymptotic behavior modulo a function with exponential decay,

$$I_0\left(s, \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} g, \chi\right) = |a/b|^{1+s} I_0^+(s,g,ab,\chi) + |a/b|^{1-s} I_0^-(s,g,ab,\chi)$$

where

$$I_0^+(s,g,\alpha,\chi) = \int_{T(F)\backslash T(\mathbb{A})} \sum_{u\in\mu_K^2\backslash F^\times} \theta_0 f(g, q(t)u\alpha)\chi(t)dt$$

$$I_0^-(s,g,\alpha,\chi) = \int_{T(F)\backslash T(\mathbb{A})} \sum_{u\in\mu_K^2\backslash F^\times} \theta_0 \widetilde{f}(g, q(t)u\alpha)\chi(t)dt.$$

It is clear that these two function vanish if χ does not factor through the the norm $\mathrm{N}_{E/F}$. If χ factors through the norm, then χ is induced from exactly two characters μ_i on $F^\times\backslash\mathbb{A}_F^\times$: $\chi(t) = \mu_1(\mathrm{N}_{E/F}t) = \mu_2(\mathrm{N}_{E/F}t)$. Then the above integral can be written as integrals over $F^\times\backslash\mathbb{A}^\times$:

$$I_0^+(s,g,\alpha,\chi) = \mu_1(\alpha)^{-1} I_0^+(s,g,\mu_1) + \mu_2(\alpha)^{-1} I_0^+(s,g,\mu_2)$$
$$I_0^-(s,g,\alpha,\chi) = \mu_1(\alpha)^{-1} I_0^-(s,g,\mu_1) + \mu_2(\alpha)^{-1} I_0^-(s,g,\mu_2)$$

where

$$I^+(s,g,\mu_i) = \frac{1}{2}\int_{F^\times\backslash\mathbb{A}^\times} \sum_{u\in\mu_K^2\backslash F^\times} \theta_0 f(g, zu)\mu_i(z)dz$$

$$I^-(s,g,\mu_i) = \frac{1}{2}\int_{F^\times\backslash\mathbb{A}^\times} \sum_{u\in\mu_K^2\backslash F^\times} \theta_0 \widetilde{f}(g, zu)\mu_i(z)dz.$$

In this way, we see that $I_0(s,g,\chi)$ is a linear combination of principal series induced from characters on the upper triangular groups:

$$\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto |a/b|^{1/2\pm s} \cdot \mu_i^{-1}(ab).$$

Let $J(s,g,\chi)$ be the Eisenstein series formed by $I_0(s,g,\chi)$:

$$J(s,g,\chi) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} I_0(s,\gamma g,\chi).$$

The constant term of $J(s,g,\chi)$ has two terms $I_0(s,g,\chi)$ and $\widetilde{I}_0(s,g,\chi)$ which is a linear combination of series in the characters:

$$\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} \mapsto |a/b|^{-1/2\pm s} \cdot \mu_i^{-1}(ab).$$

In this way, the proof of the proposition is complete. □

We now are ready to apply Proposition 3.6.1 to the form $I'(0,g,\chi) - J'(0,g,\chi)$. More precisely,

$$\mathscr{P}r'(I'(0,g,\chi)) = \mathscr{P}r'(I'(0,g,\chi) - J'(0,g,\chi)) = \mathscr{P}r(I'(0,g,\chi)) - \mathscr{P}r(J'(0,g,\chi))$$

where in the right-hand side the operator $\mathscr{P}r$ is defined after Proposition 3.6.1.

In the following we want to study $\mathscr{P}r(J'(0,g,\chi))$, which is computed from the limit of $J'_{\psi,s'}(0,g,\chi)$ as $s' \to 0$. Writing $J(s,g,\chi)$ as a linear combination of Eisenstein series $J^{\pm}(s,g,\mu_i)$ defined by functions in $I^{\pm}(s,g,\mu_i)$, we need only treat the Fourier coefficients $J^{\pm}_{\psi,s'}(s,g,\mu_i)$. From the proof of Proposition 3.6.2, we see that their Whittaker functions have a decomposition

$$J^{\pm}_{\psi}(s,g,\mu_i) = W^{(2)}(\pm s, g_\infty) J^{\pm}_{\psi}(s, g_f, \mu_i)$$

where $W^{(2)}(s,g_\infty)$ at each archimedean place has weight 2, trivial central character and value y^{1+s} at $g = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$. In this way

$$J^{\pm}_{\psi,s'}(s,g,\mu_i) = W^{(2)}(g_\infty) J^{\pm}_{\psi}(s,g_f,\mu_i) \frac{\Gamma(1+s+s')}{(4\pi)^{s+s'}}.$$

Take the limit as $s' \to 0$, and the derivative at $s=0$ to obtain

$$\mathscr{P}r(J'(0,g,\chi))_\psi := \lim_{s' \longrightarrow 0} J'_{\psi,s'}(0,g,\chi) = W^{(2)}(g_\infty) \frac{d}{ds}|_{s=0} J^{\pm}_{\psi}(s,g_f,\mu_i) \frac{\Gamma(1+s)}{(4\pi)^s}.$$

It is clear that it is a sum of Fourier coefficients of Eisenstein series and their derivations (§4.4.4 in [Zh1]). In summary, we have shown:

Proposition 3.6.3. *The difference* $\mathscr{P}r'(I'(0,g,\chi)) - \mathscr{P}r(I'(0,g,\chi))$ *is a finite sum of Eisenstein series and their derivations.*

3.7 Holomorphic kernel function

We now apply holomorphic projection formula to $I'(0,g)$ which has the following expression by Proposition 3.2.3,

$$\begin{aligned} I'(0,g) = &- \sum_{v \text{ nonsplit}} I'(0,g)(v) + 2\log\delta(g) \sum_{u \in \mu_K^2 \backslash F^\times, y \in E} r(g)\phi(y,u) \\ &- c_0 \sum_{u \in \mu_K^2 \backslash F^\times, y \in E} r(g)\phi(y,u) - \sum_v \sum_{u \in \mu_K^2 \backslash F^\times, y \in E} c_{\phi_v}(g,y,u) r(g^v)\phi^v(y,u). \end{aligned}$$

For a function on $f(g)$ on $\mathrm{GL}_2(\mathbb{A})$ which is invariant under $N(F)$ on the left, we let $f^*(g)$ denote the sum of non-constant coefficients in the Fourier coefficients under left translation by $N(F)\backslash N(\mathbb{A})$.

Proposition 3.7.1. *Modulo a finite sum of Eisenstein series and their derivation,*

$$\begin{aligned}\mathscr{P}r(I'(0,g)) = &-\sum_{v|\infty}\widetilde{I}'(0,g)(v) - \sum_{v\nmid\infty \text{ nonsplit}} I'(0,g)(v)\\ &+2\sum_{(y,u)\in\mu_K\backslash E^\times\times F^\times}(\log\delta_f(g_f)+\frac{1}{2}\log|uq(y)|_f)r(g)\phi(y,u)\\ &-\sum_v\sum_{u\in\mu_K^2\backslash F^\times, y\in E^\times} c_{\phi_v}(g,y,u)r(g^v)\phi^v(y,u) - c_1\sum_{u\in\mu_K^2\backslash F^\times, y\in E^\times} r(g)\phi(y,u).\end{aligned}$$

Here for an archimedean v,

$$\begin{aligned}\widetilde{I}'(0,g)(v) &= 2\int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\widetilde{\mathscr{K}_\phi^{(v)}}(g,(t,t))dt,\\ \widetilde{\mathscr{K}_\phi^{(v)}}(g,(t_1,t_2)) &= \sum_{a\in F^\times}\widetilde{\lim}_{s\to 0}\sum_{y\in\mu_K\backslash(B(v)_+^\times - E^\times)} r(g,(t_1,t_2))\phi(y)_a\ k_{v,s}(y),\\ k_{v,s}(y) &= \frac{\Gamma(s+1)}{2(4\pi)^s}\int_1^\infty\frac{1}{t(1-\xi_v(y)t)^{s+1}}dt,\\ c_1 &= \frac{d}{ds}|_{s=0}\left(\log\frac{L(s,\eta)}{L(s+1,\eta)}\right)+\frac{1}{2}(\gamma+\log 4\pi)[F:\mathbb{Q}].\end{aligned}$$

Other terms are the same as in Proposition 3.2.3.

Proof. We will consider the image under $\mathscr{P}r$ of each term on the right-hand side. We will see that just a few terms are changed since the operator $\mathscr{P}r$ only changes the non-holomorphic terms.

First let's look at $I'(0,g)(v)$, which is always the main term. By Proposition 3.2.1,

$$I'(0,g)(v) = 2\int_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\mathscr{K}_\phi^{(v)}(g,(t,t))dt.$$

with

$$\mathscr{K}_\phi^{(v)}(g,(t_1,t_2)) = \mathscr{K}_{r(t_1,t_2)\phi}^{(v)}(g) = \sum_{u\in\mu_K^2\backslash F^\times}\sum_{y\in B(v)-E} k_{r(t_1,t_2)\phi_v}(g,y,u)r(g,(t_1,t_2))\phi^v(y,u).$$

Note that the integral above is just a finite sum.

We have a simple rule

$$r(n(b)g,(t_1,t_2))\phi^v(y,u) = \psi(uq(y)b)\ r(g,(t_1,t_2))\phi^v(y,u),$$

and its analogue

$$k_{r(t_1,t_2)\phi_v}(n(b)g,y,u) = \psi(uq(y)b)k_{r(t_1,t_2)\phi_v}(g,y,u)$$

showed in Proposition 3.2.2. By these rules it is easy to see that the first Fourier coefficient is given by

$$\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))_\psi = \sum_{(y,u)\in\mu_K\backslash((B(v)-E)\times F^\times)_1} k_{r(t_1,t_2)\phi_v}(g_v,y_v,u_v)r(g,(t_1,t_2))\phi^v(y,u).$$

If v is non-archimedean, all the infinite components are already holomorphic of weight two. So the operator $\mathscr{P}r$ doesn't change $\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))_\psi$ at all. Thus

$$\begin{aligned}
&\mathscr{P}r(\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))) = \sum_{a\in F^\times} \mathscr{K}_\phi^{(v)}(d^*(a)g,(t_1,t_2))_\psi \\
=&\sum_{a\in F^\times}\sum_{(y,u)\in\mu_K\backslash((B(v)-E)\times F^\times)_1} k_{r(t_1,t_2)\phi_v}(d^*(a)g_v,y_v,u_v)r(d^*(a)g,(t_1,t_2))\phi^v(y,u) \\
=&\sum_{a\in F^\times}\sum_{(y,u)\in\mu_K\backslash((B(v)-E)\times F^\times)_1} k_{r(t_1,t_2)\phi_v}(g_v,ay_v,a^{-1}u_v)r(g,(t_1,t_2))\phi^v(ay,a^{-1}u) \\
=&\sum_{a\in F^\times}\sum_{(y,u)\in\mu_K\backslash((B(v)-E)\times F^\times)_a} k_{r(t_1,t_2)\phi_v}(g_v,y_v,u_v)r(g,(t_1,t_2))\phi^v(y,u) \\
=&\mathscr{K}_\phi^{(v)}(g,(t_1,t_2)).
\end{aligned}$$

Here the third equality follows from Proposition 3.2.2 that $k_{r(t_1,t_2)\phi_v}$ transforms according to the Weil representation under upper triangular matrices. We conclude that $\mathscr{P}r$ doesn't change $\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))$ if v is non-archimedean, and thus we have $\mathscr{P}r(I'(0,g)(v)) = I'(0,g)(v)$.

Now we look at the case that v is archimedean. The only difference is that we need to replace $k_{\phi_v}(g,y,u)$ by some $\widetilde{k}_{\phi_v,s}(g,y,u)$, and then take a "quasi-limit" $\widetilde{\lim}$. It suffices to consider the case that $uq(y)=1$, it is given by

$$\widetilde{k}_{\phi_v,s}(g,y,u) = 4\pi W^{(2)}(g_v)\int_{F_{v,+}} y_0^s e^{-2\pi y_0} k_{\phi_v}(d^*(y_0),y,u)\frac{dy_0}{y_0}.$$

Then $\widetilde{k}_{\phi_v,s}(g,y,u)\neq 0$ only if $u>0$, since $k_{\phi_v}(d^*(y_0),y,u)\neq 0$ only if $u>0$.

Assume that $u>0$, which is equivalent to $q(y)>0$ since we assume $uq(y)=1$ for the moment. By Proposition 3.5.1,

$$\begin{aligned}
&\int_{F_{v,+}} y_0^s e^{-2\pi y_0} k_{\phi_v}(d^*(y_0),y,u)\frac{dy_0}{y_0} = -\frac{1}{2}\int_{F_{v,+}} y_0^s e^{-2\pi y_0}\ \mathrm{Ei}(4\pi uq(y_2)y_0)\ y_0 e^{-2\pi y_0}\frac{dy_0}{y_0} \\
=&-\frac{1}{2}\int_0^\infty y_0^{s+1}e^{-4\pi y_0}\ \mathrm{Ei}(-4\pi\alpha y_0)\frac{dy_0}{y_0} \qquad (\alpha = -uq(y_2) = -\frac{q(y_2)}{q(y)}>0) \\
=&\frac{1}{2}\int_0^\infty y_0^{s+1}e^{-4\pi y_0}\int_1^\infty t^{-1}e^{-4\pi\alpha y_0 t}dt\frac{dy_0}{y_0} = \frac{1}{2}\int_1^\infty t^{-1}\int_0^\infty y_0^{s+1}e^{-4\pi(1+\alpha t)y_0}\frac{dy_0}{y_0}dt \\
=&\frac{\Gamma(s+1)}{2(4\pi)^{s+1}}\int_1^\infty \frac{1}{t(1+\alpha t)^{s+1}}dt.
\end{aligned}$$

Hence,

$$\widetilde{k}_{\phi_v,s}(g,y,u) = W^{(2)}(g_v)\frac{\Gamma(s+1)}{2(4\pi)^s}\int_1^\infty \frac{1}{t(1-\frac{q(y_2)}{q(y)}t)^{s+1}}dt = W^{(2)}(g_v)k_{v,s}(y).$$

This matches the result in the proposition. Since $k_{v,s}(y)$ is invariant under the multiplication action of $F^\times$ on y, it is easy to get

$$\mathscr{P}r(\mathscr{K}_\phi^{(v)}(g,(t_1,t_2))) = \widetilde{\mathscr{K}_\phi^{(v)}}(g,(t_1,t_2)), \qquad \mathscr{P}r(I'(0,g)(v)) = \widetilde{I}'(0,g)(v).$$

Similarly, we have

$$\mathscr{P}r\left(\sum_{u\in\mu_K^2\backslash F^\times, y\in E} r(g)\phi(y,u)\right) = \sum_{u\in\mu_K^2\backslash F^\times, y\in E^\times} r(g)\phi(y,u),$$

$$\mathscr{P}r\left(\sum_{u\in\mu_K^2\backslash F^\times, y\in E} c_{\phi_v}(g,y,u)r(g^v)\phi^v(y,u)\right) = \sum_{u\in\mu_K^2\backslash F^\times, y\in E^\times} c_{\phi_v}(g,y,u)r(g^v)\phi^v(y,u).$$

It remains to take care of

$$\log\delta(g)\sum_{u\in\mu_K^2\backslash F^\times, y\in E} r(g)\phi(y,u).$$

Its first Fourier coefficient is just

$$\begin{aligned}&\log\delta(g)\sum_{(y,u)\in\mu_K\backslash(E^\times\times F^\times)_1} r(g)\phi(y,u)\\ =&\sum_{(y,u)\in\mu_K\backslash(E^\times\times F^\times)_1}\log\delta(g_f)r(g)\phi(y,u) + \sum_{(y,u)\in\mu_K\backslash(E^\times\times F^\times)_1} r(g)\phi_f(y,u)\cdot\log\delta(g_\infty)W^{(2)}(g_\infty).\end{aligned}$$

Then $\mathscr{P}r$ doesn't change the first sum of the right-hand side, but changes $\log\delta(g_\infty)W^{(2)}(g_\infty)$ in the second sum to some multiple $c_2\,W^{(2)}(g_\infty) = c_2\,r(g)\phi_\infty(y,u)$, where c_2 is some constant. Hence we see that

$$\begin{aligned}&\mathscr{P}r\left(\log\delta(g)\sum_{u\in\mu_K^2\backslash F^\times, y\in E} r(g)\phi(y,u)\right)^*\\ =&\sum_{a\in F^\times}\sum_{(y,u)\in\mu_K\backslash(E^\times\times F^\times)_1}\log\delta(d^*(a)g_f)r(d^*(a)g)\phi(y,u)\\ &+c_2\sum_{a\in F^\times}\sum_{(y,u)\in\mu_K\backslash(E^\times\times F^\times)_1} r(d^*(a)g)\phi(y,u)\\ =&\sum_{(y,u)\in\mu_K\backslash E^\times\times F^\times}(\log\delta(g_f)+\log|uq(y)|^{\frac{1}{2}})r(g)\phi(y,u) + c_2\sum_{(y,u)\in\mu_K\backslash E^\times\times F^\times} r(g)\phi(y,u).\end{aligned}$$

As for the constant, we have

$$\frac{c_2}{[F:\mathbb{Q}]} = 4\pi \lim_{s\to 0} \int_{F_{v,+}} y^s e^{-2\pi y} \ \log |y|^{\frac{1}{2}} e^{-2\pi y} \frac{dy}{y} = 2\pi \int_0^\infty e^{-4\pi y} \log y dy = -\frac{1}{2}(\gamma + \log 4\pi).$$

□

4 Shimura curves, Hecke operators, CM-points

In this section, we will review the theory of Shimura curves, CM-points, generating series of Hecke operators, and a preliminary decomposition of height pairing of CM-points using Arakelov theory.

In §4.1, we will describe a projective system of Shimura curves X_U for a totally definite incoherent quaternion algebra $\mathbb{B}$ over a totally real field F indexed by compact open subgroups U of $\mathbb{B}_f^\times$. In §4.2, we will define a generating function $Z_\phi(g)$ with coefficients in $\mathrm{Pic}(X_U \times X_U)$ for each $\phi \in \mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)$, and will show that it is automorphic in $g \in \mathrm{GL}_2(\mathbb{A})$. This series is an extension of Kudla's generating series for Shimura varieties of orthogonal type [Ku1]. In this case, the modularity of its cohomology class is proved by Kudla-Millson [KM1, KM2, KM3]. The modularity as Chow cycles are proved in our previous work [YZZ]. In §4.3, for E an imaginary quadratic extension of F, we define a set of CM-points by E bijective to $E^\times\backslash\mathbb{B}_f^\times$. For two CM-points represented by $\beta_i \in \mathbb{B}_f^\times$, we define a function $Z_\phi(g, \beta_1, \beta_2)$ for $g \in \mathrm{GL}_2(\mathbb{A})$ by means of height pairing. It can be viewed as a function in $g \in \mathrm{GL}_2(\mathbb{A})$ and $(\beta_1, \beta_2) \in \mathrm{GO}(\mathbb{A})$ compatible with Weil representation on $\mathscr{S}(\mathbb{V} \times \mathbb{A}^\times)_{\mathrm{GO}(F_\infty)}$. This function is automorphic for g and left invariant under the diagonal action of $\mathbb{A}_E^\times$ on (β_1, β_2). In §4.4 and §4.5, using Arakelov theory, we will decompose $Z_\phi(g, t_1, t_2)$ into a finite sum of the form $\langle Z_\phi(g)\widehat{\xi}, \widehat{\eta}\rangle$ where $\widehat{\xi}$ is the arithmetic Hodge class, a theta series, and an infinite sum of local pairings:

$$\langle Z_\phi^* t_1, t_2\rangle = \sum_v j_v(Z_\phi^*(g)t_1, t_2)\log N_v + i(Z_\phi^* t_1, t_2).$$

In §4.6, we show that $\langle Z_\phi(g)\widehat{\xi}, \widehat{\eta}\rangle$ is equal to a finite sum of Eisenstein series and their derivations in the sense of §4.4.4 in our previous paper [Zh1]. The computation of the local pairing is the main content in the next section.

4.1 Shimura curves

In the following, we will review the theory of Shimura curves following our previous paper [Zh2]. Let F be a totally real number field. Let $\mathbb{B}$ be a quaternion algebra over $\mathbb{A}$ with odd ramification set Σ including all archimedean places. Then for each open subset U of $\mathbb{B}_f^\times$ we have a Shimura curve X_U. The curve is not connected; its set of connected components is can be parameterized by $F_+^\times\backslash\mathbb{A}_f^\times/q(U)$. For each archimedean place τ of F, the set of complex points at τ is given as follows:

$$X_{U,\tau}(\mathbb{C}) = B(\tau)_+^\times\backslash\mathscr{H} \times \mathbb{B}_f^\times/U \cup \{\mathrm{Cusps}\}$$

where $B(\tau)_+^\times$ is the group of totally positive elements in a quaternion algebra $B(\tau)$ over F with ramification set $\Sigma \setminus \{\tau\}$ with an action on $\mathscr{H}^\pm$ by some fixed isomorphisms

$$B(\tau) \otimes_\tau \mathbb{R} = M_2(\mathbb{R})$$

$$B(\tau) \otimes \mathbb{A}_f \simeq \mathbb{B}_f,$$

and where $\{\text{Cusp}\}$ is the set of cusps which is non-empty only when $F = \mathbb{Q}$ and $\mathbb{B}_f = M_2(\mathbb{A}_f)$.

For two open compact subgroups $U_1 \subset U_2$ of $\mathbb{B}_f^\times$, one has a canonical morphism π_{U_1,U_2} : $X_{U_1} \longrightarrow X_{U_2}$ which satisfies the composition property. Thus we have a projective system X of curves X_U. For any $x \in \mathbb{B}_f$, we also have isomorphism T_x : $X_U \longrightarrow X_{x^{-1}Ux}$ which induces an automorphism on the projective system X and compatible with multiplication on $\mathbb{B}_f^\times$: $T_{xy} = T_x \cdot T_y$. All of these morphisms on X_U's has obvious description on complex manifolds $X_{U,\tau}(\mathbb{C})$. The induced actions are the obvious one on the sets of connected components after taking norm of U_i and x.

An important tool to study Shimura curves is to use modular interpretation. For a fixed archimedean place τ, the space $\mathscr{H}^\pm$ parameterizes Hodge structures on $V_0 := B(\tau)$ which has type $(-1,0) + (0,-1)$ (resp $(0,0)$) on $V_0 \otimes_\tau \mathbb{R}$ (resp. $V_0 \otimes_\sigma \mathbb{R}$ for other archimedean places $\sigma \neq \tau$). The non-cuspidal part of $X_{U,\tau}(\mathbb{C})$ parameterizes Hodge structure and level structures on a $B(\tau)$-module V of rank 1.

Due to the appearance of type $(0,0)$, the curve X_U does not parameterize abelian varieties unless $F = \mathbb{Q}$. To get a modular interpretation, we use an auxiliary imaginary quadratic extension K over F with complex embeddings $\sigma_K : K \longrightarrow \mathbb{C}$ for each archimedean places σ of F other than τ. These σ_K's induce a Hodge structure on K which has type $(0,0)$ on $K \otimes_\tau \mathbb{R}$ and type $(-1,0)+(0,-1)$ on $K \otimes_\sigma \mathbb{R}$ for all $\sigma \neq \tau$. Now the tensor product of Hodge structures on $V_K := V \otimes_F K$ is of type $(-1,0)+(0,-1)$. In this way, X_U parameterizes some abelian varieties with homology group H_1 isomorphic to V_K. The construction makes X_U a curve over the reflex field for σ_K's:

$$K^\sharp = \mathbb{Q}\left(\sum_{\sigma \neq \tau} \sigma(x), \quad x \in K\right).$$

See our paper [Zh1] for a construction following Carayol in the case $K = F(\sqrt{d})$ with $d \in \mathbb{Q}$ where $\sigma_K(\sqrt{d})$ is chosen independent of σ.

The curve X_U has a canonical integral model $\mathscr{X}_U$ over O_F. At each finite place v of F, the base change $X_{U,v}$ over O_{F_v} will parameterizes some p-divisible group with some level structure. Locally at a geometric point, $X_{U,v}$ is the universal deformation of the represented p-divisible group.

Hodge class and Neron–Tate height pairing

The curve X_U has a *Hodge class* $\mathscr{L}_U \in \mathrm{Pic}(X_U) \otimes \mathbb{Q}$ which is compatible with pull-back morphism and such that $\mathscr{L}_U \simeq \omega_{X_U}$ when U is sufficiently small.

We also define a class $\xi_U \in \mathrm{Pic}(X_U) \otimes \mathbb{Q}$ which has degree 1 on each connected component and is proportional to $\mathscr{L}_U$. One can use ξ to define an projection $\mathrm{Div}(X_U) \longrightarrow \mathrm{Div}^0(X_U)$ by sending D to $D - \deg(D)\xi_U$ where $\deg D$ is the degree function on the connected components $\{X_i\}$ of X_U: $\deg D(X_i) := \deg(D|_{X_i})$. The class $\deg D \cdot \xi_U$ has restriction $\deg(D|_{X_i})\xi_{X_i}$ on X_i.

The Jacobian variety J_U of X_U is defined to be the variety over F to represent line bundles on X_U (or base change) of degree 0 on every connected components X_i. Over an extension of F, J_U is the product of Jacobian varieties J_i of the connected components X_i. The Neron–Tate height $\langle\cdot,\cdot\rangle$ on $J_U(\bar F)$ is defined using the Poincare divisor on $J_i\times J_i$.

4.2 Hecke correspondences and generating series

We want to define some correspondences on X_U, i.e., some divisor classes on $X_U\times X_U$. The projective system of surfaces $X_U\times X_U$ has an action by $\mathbb{B}_f^\times\times\mathbb{B}_f^\times$. Let K denote the open compact subgroup $K=U\times U$.

Hecke operators

For any double coset UxU of $U\backslash\mathbb{B}_f^\times/U$, we have a Hecke correspondence

$$Z(x)_U\in Z^1(X_U\times X_U)$$

defined as the image of the morphism

$$(\pi_{U\cap xUx^{-1},U},\pi_{U\cap x^{-1}Ux,U}\circ T_x):\qquad Z_{U\cap xUx^{-1}}\longrightarrow X_U^2.$$

In terms of complex points at a place of F as above, the Hecke correspondence $Z(x)_U$ takes

$$(z,g)\longrightarrow\sum_i(z,gx_i)$$

for points on $X_{U,\tau}(\mathbb{C})$ represented by $(z,g)\in\mathscr{H}^\pm\times\mathbb{B}_f$ where x_i are representatives of UxU/U.

Hodge class

On $M_K:=X_U\times X_U$, one has a *Hodge bundle* $\mathscr{L}_K\in\mathrm{Pic}(M_K)\otimes\mathbb{Q}$ defined as

$$\mathscr{L}_K=\frac{1}{2}(p_1^*\mathscr{L}_U+p_2^*\mathscr{L}_U).$$

Generating Function

Let $\mathbb{V}$ denote the orthogonal space $\mathbb{B}$ with quadratic form q. Let $\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)_{\mathrm{GO}(V_\infty)}$ denote the space $\mathscr{S}(V_\infty,F_\infty^\times)^{\mathrm{GO}(F_\infty)}\otimes\mathscr{S}(V_{\mathbb{A}_f}\times\mathbb{A}_f^\times)$ which is isomorphic to the maximal quotient of $\mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)$ via integration on $\mathrm{GO}(F_\infty)$.

For any $(x,u)\in\mathbb{V}\times\mathbb{A}^\times$, let us define a cycle $Z(x,u)_K$ on $X_U\times X_U$ as follows. This cycle is non-vanishing only if $q(x)u\in F^\times$ or $x=0$. If $q(x)u\in F^\times$, then we define $Z(x,u)_K$ to be the Hecke operator UxU defined in the last subsection. If $x=0$, then we define $Z(x,u)_K$ to be the push-forward of the Hodge class on the subvariety M_α which is union of connected components $X_\alpha\times X_\beta$ with $\alpha,\beta\in F^\times\backslash\mathbb{A}^\times/F_{\infty,+}^\times\nu(U)$ such that $u=\alpha\beta\mod F_+^\times q(U)F_{\infty,+}^\times$.

For $\phi \in \mathscr{S}(\mathbb{V}\times\mathbb{A}^\times)_{\mathrm{GO}(F_\infty)}$ which is invariant under $K\cdot\mathrm{GO}(F_\infty)$, we can form a generating series

$$Z_\phi(g) = \sum_{(x,u)\in(K\cdot\mathrm{GO}(F_\infty))\backslash\mathbb{V}\times\mathbb{A}^\times} r(g)\phi(x,u)Z(x,u)_K.$$

It is easy to see that this definition is compatible with pull-back maps in Chow groups in the projection $M_{K_1}\longrightarrow M_{K_2}$ with $K_i = U_i \times U_i$ and $U_1 \subset U_2$. Thus it defines an element in the direct limit $\mathrm{Ch}^1(M)_\mathbb{Q} := \lim_K \mathrm{Ch}^1(M_K)$. For any $h \in (\mathbb{B}_f^\times)^2$, let $\rho(h)$ denote the pull-back morphism on $\mathrm{Ch}^1(M)$ by right translation of h. Then it is easy to verify

$$Z_{r(h)\phi} = \rho(h)Z_\phi.$$

Proposition 4.2.1. *The series Z_ϕ is absolutely convergent and defines an automorphic form for* $\mathrm{GL}_2(\mathbb{A})$.

We will reduce the proposition to the modularity proved in [YZZ].

Let M_K^1 be the subvariety of the union of connected components $X_\alpha \times X_\beta$ with $\alpha, \beta \in F_+^\times\backslash\mathbb{A}_f^\times/\nu(U)$ such that $\alpha \in F_+^\times\nu(U)$. Then M_K^1 is a Shimura variety of orthogonal type associated to the subgroup $\mathrm{GSpin}(\mathbb{V})$ of $\mathbb{B}^\times \times \mathbb{B}^\times$ of pairs (g_1, g_2) with same norm. For any element $\phi^1 \in \mathscr{S}(\mathbb{V})_{O(F_\infty)}$ then one can define the generating series

$$Z_{\phi^1}(g) = \sum_{x\in K^1\cdot O(F_\infty)\backslash\mathbb{V}} r(g)\phi^1(x)Z(x)_{K^1}$$

where $K^1 = K\cap\mathrm{GSpin}(\mathbb{V}_f)$ and $Z(x)_{K^1}$ is non-zero only if $q(x) \in F^\times$ or $x = 0$. If $q(x) \in F^\times$, then $Z(x)_{K^1}$ is defined as Heck operator UxU; if $x = 0$, then $Z(x)_{K^1}$ is defined as $c_1(\mathscr{L}_{K^1}^\vee)$. In [YZZ], we have shown that $Z_{\phi^1}(g)$ is absolutely convergent and defines an automorphic form on $\mathrm{SL}_2(\mathbb{A})$.

For any $h \in \mathrm{GO}(\widehat{V})$, let i_h denote the composition of the embedding $M^1\longrightarrow M$ and the translation T_h on M. Then we can show that M_K is covered by $i(M_{K^h}^1)$ for h runs through a set of elements in $\mathrm{GO}(\widehat{V}) = \mathbb{B}_f^\times \times \mathbb{B}_f^\times$ whose similitudes represents the cosets $F_+^\times\backslash\mathbb{A}_f^\times/\nu(K)$, and $K^h = \mathrm{GSpin}(\widehat{V}) \cap hKh^{-1}$. Thus it is clear that

$$Z_\phi = \sum_h i_{h*}i_h^*Z_\phi.$$

Let us compute $i_h^*Z_\phi$. By definition,

$$i_h^*Z_\phi(g) = \sum_{(x,u)\in K\cdot\mathrm{GO}(F_\infty)\backslash\mathbb{V}\times\mathbb{A}^\times} r(g)\phi(x,u)i^*Z(x,u)_K$$

If $x = 0$, then $i_h^*Z(x,u) \neq 0$ only if $u \in q(h)F_+^\times\nu(K)$ in which case it is given as $Z(0)_{K^h}$; if $q(x)u \in F^\times$, then $i^*Z(x,u)_K \neq 0$ only if $\nu(h)q(x) \in F_+^\times\nu(K)$ in which case it is given by $Z(y)_{K^h}$, where $y \in Kx$ with norm in $F_+^\times$. In this way the sum above becomes

$$i_h^*Z_{r(g)\phi} = \sum_{u\in\mu_K'\backslash F_+^\times}\ \sum_{x\in K^hO(F_\infty)\backslash\mathbb{V}} r(g,h)\phi(x,u)Z(y)_{K^h} = \sum_{u\in\mu_K'\backslash F_+^\times} Z_{r(g,h)\phi(\cdot,u)}.$$

where $\mu'_K = F_+^\times \cap \nu(K)$. We have thus shown that Z_ϕ is a finite sum of generating series for M^1. It follows that $Z_\phi(g)$ is invariant under left translation by elements in $\mathrm{SL}_2(F)$ and is absolutely convergent. By definition, it is clear that Z_ϕ is also invariant under left translation by elements of the form $d(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$. Thus Z_ϕ is invariant under left translation by $\mathrm{GL}_2(F)$.

4.3 CM-points and height series

Let E be an imaginary quadratic extension of F with an embedding $E(\mathbb{A}_f) \subset \mathbb{B}_f$. Then $X_U(\bar{F})$ has a set of CM-points defined over E^{ab}, the maximal abelian extension of E. The set CM_U is stable under action of $\mathrm{Gal}(E^{\mathrm{ab}}/F)$ and Hecke operators. More precisely, we have a projective system of bijections

$$\mathrm{CM}_U \simeq E^\times \backslash \mathbb{B}_f^\times / U \qquad (4.3.1)$$

which is compatible with action of Hecke operators and such that the Galois action is given as follows: the action of $\mathrm{Gal}(E^{\mathrm{ab}}/E)$ acts by right multiplication of elements of $E^\times(\mathbb{A}_f)$ via the reciprocity law in class field theory:

$$E^\times \backslash E^\times(\mathbb{A}_f) \longrightarrow \mathrm{Gal}(E^{\mathrm{ab}}/E).$$

Such a bijection is unique up to left multiplication by elements in $E^\times(\mathbb{A}_f)$.

If τ is a real place of F, then the set C_U can be described as a subset of

$$X_{U,\tau}(\mathbb{C}) = B(\tau)^\times \backslash \mathscr{H}^\pm \times \mathbb{B}_f^\times / U \cup \{\text{cusps}\}$$

as the set of points

$$\mathrm{CM}_U = B(\tau)^\times \backslash B(\tau)^\times z_0 \times \mathbb{B}_f^\times / U \cup \{\text{cusps}\} \simeq E^\times \backslash \mathbb{B}_f^\times / U$$

where E is embedded into $B(\tau)$ compatible with isomorphism $B(\tau)_{\mathbb{A}_f} = \mathbb{B}_f$ and $z_0 \in \mathscr{H}$ is unique fixed point of $E^\times$.

For any $\phi \in \mathscr{S}(\mathbb{B}_\mathbb{A} \times \mathbb{A}^\times)^{U\times U}_{\mathrm{GO}(F_\infty)}$, we define the Neron–Tate height pairing

$$Z_\phi(g, \beta_1, \beta_2) \in \langle Z_\phi(g)([\beta_1]_U - \deg([\beta_1]_U)\xi), \quad [\beta_2]_U - \deg([\beta_2]_U)\xi\rangle_U, \qquad \beta_1, \beta_2 \in \mathbb{B}_f^\times.$$

Using the projection formula of height pairing, we see that this definition does not depend on the choice of U. Also this definition does not depend on the choice of the bijection 4.3.1 since the height pairing is invariant under Galois action. It follows that

$$Z_\phi(g, t\beta_1, t\beta_2) = Z_\phi(g, \beta_1, \beta_2).$$

In this way, we may view this as a function on $\mathrm{GO}(\mathbb{V})$ through projection

$$\mathrm{GO}(\mathbb{V}) = \Delta(\mathbb{A}) \backslash \mathbb{B}^\times \times \mathbb{B}^\times \longrightarrow \mathrm{GO}(\mathbb{V}_f) = \Delta(\mathbb{A}_f) \backslash \mathbb{B}_f^\times \times \mathbb{B}_f^\times$$

Using the projection formula and the formula $Z_{r(h)\phi} = \rho(h)Z_\phi(g)$ where $\rho(h)$ is the pull-back morphism of right translation by $h \in \mathrm{GO}(\mathbb{V})$, one can show that the resulting function $Z_\phi(g, h)$ is equivariant under Weil representation:

$$Z_{r(g_1,h_1)\phi}(g_2, h_2) = Z_\phi(g_2g_1, h_2h_1), \qquad g_i \in \mathrm{GL}_2(\mathbb{A}), \quad h_i \in \mathrm{GO}(\mathbb{V}).$$

This function is automorphic for the first variable and invariant for the second variable under left diagonal multiplication by $\mathbb{A}_E^\times$.

Let T denote $E^\times$ as an algebraic group over F. Any $\beta \in \mathbb{B}_f^\times$ gives a CM-point in CM_U which is denoted by $[\beta]_U$ or just β if U is clear. We are particularly interested in the case that $\beta \in T(\mathbb{A}_f)$, i.e. CM-points that are in the image in X_U of the zero-dimensional Shimura variety

$$C_U = T(F)\backslash T(\mathbb{A}_f)/U_T$$

associated to $T = E^\times$. Here $U_T = U \cap T(\mathbb{A}_f)$. Notice that the set C_U does not depend on the choice of bijection 4.3.1. For a finite character χ of $T(F)\backslash T(\mathbb{A})$, we can define

$$Z_\phi(g, \chi) = \int_{(T(F)\backslash T(\mathbb{A}_f))^2} Z_\phi(g, t_1, t_2)\chi(t_1t_2^{-1})dt_1dt_2.$$

4.4 Arithmetic intersection pairing

The Hodge bundle $\mathscr{L}_U$ can be extended into a metrized line bundle on $\mathscr{X}_U$. More precisely, at an archimedean place, the metric can defined by using Hodge structure or equivalently normalized such that its pull-back on $\Omega^1_{\mathscr{H}}$ takes the form:

$$\|f(z)dz\| = 4\pi \cdot \mathrm{Im} z \cdot |f(z)|.$$

At a finite place, we may take an extension $\mathscr{L}_{U,v}$ by using the fact that $\mathscr{L}_U$ is twice of the cotangent bundle of the divisible groups. The resulting bundle on $\mathscr{X}_U$ with metrics at archimedean places is denoted by $\widehat{\mathscr{L}_U}$. Also we have an arithmetic class $\widehat{\xi}_U$ induced from $\widehat{\mathscr{L}_U}$.

The Hodge index theorem on the Jacobian J_i of geometric connected components X_i of X implies a Hodge index theorem on X_U: for any two divisors D_1 and D_2 of degree 0 on every connected component. Then their Neron–Tate height pairing is given by the following formula:

$$\langle D_1, D_2 \rangle = -\widehat{D}_1 \cdot \widehat{D}_2$$

where $\widehat{D}_i$ are some "*flat*" arithmetic divisors extending D_i. Here, $\widehat{D}$ is flat in the sense that its curvature at archimedean places and its intersection with vertical divisors are all 0.

For the computation in the next section, we will consider the group $\widehat{\mathrm{Pic}}(X_U)_\xi$ of arithmetic line bundles $\widehat{\mathscr{L}}$ such that $\widehat{c}_1(\widehat{\mathscr{L}}) - \deg D \cdot \widehat{c}_1(\widehat{\xi})$ is flat, we call such that line bundle $\widehat{\xi}$-admissible. Moreover for any divisor D, there is a unique extension $\widehat{D}$ which is $\widehat{\xi}$-admissible such that the vertical component in every fiber of $\mathscr{X}_U$ has zero intersection with $\widehat{\xi}$. Such a construction is totally local and compatible with pull-back morphism for the projections

$X_{U_1}\longrightarrow X_{U_2}$ and Hecke operators. In particular, we have a well defined Green's function $g_v(x,y)$ on $X_v(\bar F_v)$ such that for any two distinct points $x, y \in X_U(K)$,

$$\widehat{x}\cdot\widehat{y} = -\frac{1}{[K:F]}\sum_v \sum_{\sigma:K\longrightarrow \bar F_v} g_v(\sigma(x),\sigma(y)).$$

On $\mathscr{X}_U$, one has a decomposition

$$g_v(x,y) = i_v(x,y) + j_v(x,y)$$

where $i_v(x,y)$ is the intersection of the Zariski closures of x and y and $j_v(x,y)$ is a locally constant function on $X_{U,v}(\bar F_v)\times X_{U,v}(\bar F_v)$.

4.5 Decomposition of the height pairing

Our goal in the geometric side is to compute

$$Z_\phi(g,t_1,t_2) = \langle Z_\phi(g)(t_1-\deg(t_1)\xi), t_2-\deg(t_2)\xi\rangle, \qquad t_1,t_2\in C_U.$$

Once ξ is extended to an arithmetic class $\hat\xi$, we get the decomposition:

$$\langle Z_\phi(g)t_1,t_2\rangle - \langle Z_\phi(g)t_1,\deg(t_2)\xi\rangle - \langle Z_\phi(g)\deg(t_1)\xi,t_2\rangle + \langle Z_\phi(g)\deg(t_1)\xi,\deg(t_2)\xi\rangle.$$

We will take care of the last three terms, but let us first consider $\langle Z_\phi(g)t_1,t_2\rangle$ which is the main term. We start with some general discussion on decomposition of intersection of CM-points.

Decomposition of the height pairing

Now we want to write the height pairing into a sum of local terms in the case that $\beta_1\neq\beta_2\in \mathbb{B}_f^\times$, and get a similar decomposition even in the case $\beta_1=\beta_2$ with an"error term" $i_0(\beta,\beta)$.

We can always write

$$\langle\beta_1,\beta_2\rangle = i(\beta_1,\beta_2)+j(\beta_1,\beta_2)$$

as in [Zh2].

We first assume that $\beta_1\neq\beta_2$. Then we have decompositions

$$i(\beta_1,\beta_2) = \sum_{v\in\Sigma_F} i_v(\beta_1,\beta_2)\log N_v, \qquad j(\beta_1,\beta_2) = \sum_{v\in\Sigma_F} j_v(\beta_1,\beta_2)\log N_v$$

with

$$i_v(\beta_1,\beta_2) = \frac{1}{\#\Sigma_{E_v}}\sum_{w\in\Sigma_{E_v}} i_w(\beta_1,\beta_2), \qquad j_v(\beta_1,\beta_2) = \frac{1}{\#\Sigma_{E_v}}\sum_{w\in\Sigma_{E_v}} j_w(\beta_1,\beta_2).$$

Here Σ_{E_v} denotes the set of places of E lying over v.

To compute these local pairings, we will use:

$$i_w(\beta_1,\beta_2) = \frac{1}{[L:E]}\sum_{\sigma:L\hookrightarrow \overline{E}_w} i_{\bar w}(\beta_1^\sigma,\beta_2^\sigma) = \fint_{T(F)\backslash T(\mathbb{A}_f)} i_{\bar w}(t\beta_1,t\beta_2)dt,$$
$$j_w(\beta_1,\beta_2) = \frac{1}{[L:E]}\sum_{\sigma:L\hookrightarrow \overline{E}_w} j_{\bar w}(\beta_1^\sigma,\beta_2^\sigma) = \fint_{T(F)\backslash T(\mathbb{A}_f)} j_{\bar w}(t\beta_1,t\beta_2)dt.$$

Here L is any Galois extension of E that contains the residue fields of β_1,β_2, and the integral on $T(F)\backslash T(\mathbb{A}_f)$ takes the Haar measure with total volume one.

The pairing j_w is zero identically if w is archimedean or $X_{U,E}$ has good reduction at w. Furthermore, all the decompositions above for j still makes sense even if $\beta_1=\beta_2$.

Now we consider the self-intersection $i(\beta,\beta)$ for any $\beta\in \mathrm{CM}_U$. In next section, we will have an extended definition of $i_{\bar w}(\beta,\beta)$ for each place w of E. Then formally, we still define $i_w(\beta,\beta)$ and $i_v(\beta,\beta)$ by the above averaging formulas. The difference

$$i_0(\beta,\beta) := i(\beta,\beta) - \sum_{v\in\Sigma_F} i_v(\beta,\beta)\log N_v$$

is not zero any more. We will see that it is essentially the Faltings height of β by an arithmetic adjunction formula. But for the Gross-Zagier formula, it is enough to use the property that $i_0(\beta,\beta)$ depends only on the Galois orbit of β.

Decomposition of the kernel Function

Rewrite $Z_\phi(g)$ according to $a=q(x)u\in F$ to obtain

$$Z_\phi(g) = Z_{\phi,0}(g) + \sum_{a\in F^\times}\sum_{x\in K\backslash \mathbb{B}_f^\times} r(g)\phi(x)_a Z(x)_U$$

where

$$Z_{\phi,0}(g) = \sum_{u\in \nu(K)F_\infty^\times\backslash\mathbb{A}^\times} r(g)\phi(0,u)[\mathscr{L}_u^{-1}],$$

and

$$\phi(x)_a = \phi(x,q(x)^{-1}a).$$

The Hecke operator $Z(x)_U$ corresponds to the double coset UxU, i.e.,

$$Z(x)_U t_1 = \sum_j t_1\alpha_j, \quad \text{if } \ UxU = \coprod_j \alpha_j U.$$

It follows that

$$\begin{aligned} Z_\phi(g)[t_1] &= Z_{\phi,0}(g)[t_1] + \sum_{a\in F^\times}\sum_{x\in K\backslash \mathbb{B}_f^\times} r(g)\phi(x)_a Z(x)_U[t_1] \\ &= Z_{\phi,0}(g)[t_1] + \sum_{a\in F^\times}\sum_{x\in \mathbb{B}_f^\times/U} r(g)\phi(x)_a[t_1x]. \end{aligned}$$

Since $[t_1x] = [t_2]$ in CM_U if and only if $x \in t_1^{-1}t_2T(F)U$, the part of self-intersection comes from

$$\begin{aligned} R_\phi(g)[t_1] &= \sum_{a\in F^\times}\sum_{x\in t_1^{-1}t_2T(F)U/U} r(g)\phi(x)_a[t_1x] \\ &= \sum_{a\in F^\times}\sum_{y\in E^\times/(E^\times\cap U)} r(g)\phi(t_1^{-1}yt_2)_a[t_2] \\ &= \frac{1}{[E^\times\cap U:\mu_U]}\sum_{a\in F^\times}\sum_{y\in E^\times/\mu_U} r(g,(t_1,t_2))\phi(y)_a[t_2]. \end{aligned}$$

The index $[E^\times \cap U : \mu_U] = 1$ when U is small enough. In the following, for simplicity we always assume that $E^\times \cap U = \mu_U$.

Now we consider the decomposition of the global height $\langle Z_\phi(g)t_1, t_2\rangle$. We first write

$$\begin{aligned} \langle Z_\phi(g)t_1,t_2\rangle &= \langle Z_{\phi,0}(g)t_1,t_2\rangle + \langle Z_\phi^*(g)t_1,t_2\rangle \\ &= \langle Z_{\phi,0}(g)t_1,t_2\rangle + \sum_v j_v(Z_\phi^*(g)t_1,t_2)\log N_v + i(Z_\phi^*(g)t_1,t_2), \end{aligned}$$

where

$$\begin{aligned} Z_\phi^*(g) &= Z_\phi(g) - Z_{\phi,0}(g) = \sum_{a\in F^\times}\sum_{x\in K\backslash\mathbb{B}_f^\times} r(g)\phi(x)_a Z(x)_U, \\ Z_\phi^*(g)[t_1] &= \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a[t_1x]. \end{aligned}$$

Then

$$\begin{aligned} & i(Z_\phi^*(g)t_1,t_2) \\ = & \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a\ i(t_1x,t_2) \\ = & \sum_{a\in F^\times}\sum_{y\in E^\times/\mu_U} r(g,(t_1,t_2))\phi(y)_a i_0(t_2,t_2) + \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a\sum_v i_v(t_1x,t_2)\log N_v \\ = & \sum_v i_v(Z_\phi^*(g)t_1,t_2)\log N_v + i_0(1,1)\sum_{a\in F^\times}\sum_{y\in E^\times/\mu_U} r(g,(t_1,t_2))\phi(y)_a. \end{aligned}$$

Here we denote

$$i_v(Z_\phi^*(g)t_1,t_2) = \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a\sum_v i_v(t_1x,t_2).$$

And we have $i_0(t_2,t_2) = i_0(1,1)$ since $[1]$ and $[t]$ are Galois conjugate CM points.

4.6 Hecke action on arithmetic Hodge classes

In this subsection, we want to discuss the action of Hecke operator Z_ϕ on the arithmetic classes. More precisely, we want to study the pairing

$$\langle Z_\phi \widehat{\xi}_f, \widehat{\eta} \rangle$$

where f is a function on the set of connected components and $\widehat{\eta}$ is an arithmetic divisor class. Recall using the usual uniformization at an archimedean place, the set of connected components can be indexed by a group

$$F_+^\times \backslash \mathbb{A}_f^\times / \nu(U).$$

The function on this set is generated by characters ω. Thus it suffices to treat the case where $f = \omega$ is a character.

The action of Z_ϕ on the geometric cycle ξ_ω is given by the action on ω, the function on the set of connected components. By a result of Kudla [Ku1], the ratio of $Z_\phi^*(\xi_\omega)$ over ξ_ω is given by an Eisenstein series $E(g, \phi, \omega)$ with central character ω^{-2}:

$$Z_\phi^*(g)\xi_\omega = E^*(g, \phi, \omega)\xi_\omega.$$

The Whittaker function of this Eisenstein series is given by

$$E_\psi(g, \phi, \omega) = \sum_{x \in K\mathrm{GO}(F_\infty)\backslash \mathbb{V}^\times} r(g)\phi(x, q(x)^{-1}) Z(x, \omega)_K$$

where

$$Z(x, \omega)_K = \int_{Z(x)_K} \omega(h) dh.$$

It follows that the action on the arithmetic cycle $\widehat{\xi}$ will have formula

$$Z_\phi^*(g)\widehat{\xi_\omega} = E^*(g, \phi, \omega)\widehat{\xi_\omega} + V$$

where V is a vertical cycle. In the following we want to discuss in more details the Whittaker coefficient function of $Z_\phi(g)$.

Assume that $\phi = \otimes \phi_v$. By definition, the Whittaker function of $Z_\phi(g)$ is given by

$$Z_{\phi,\psi}(g) = \prod_v Z_{\phi_v, \psi_v}(g_v)$$

where

$$Z_v(g_v) = \sum_{x \in K_v \backslash \mathbb{V}_v^\times} r(g_v)\phi_v(x_v, q(x_v)^{-1}) Z(x_v)_{K_v}.$$

Here $K_v = \mathrm{GO}(F_v)$ for $v \mid \infty$. Notice that for almost all v, $Z_v(g_v)$ is a trivial operator. Indeed, for almost all v, $g_v \in \mathrm{GL}_2(O_v)$, ϕ_v is the characteristic function of $O_{B_v} \times O_v^\times$ where

O_{B_v} is a maximal order of B_v, and $K_v = O_{B_v}^\times \times O_{B_v}^\times$. Thus the sum $Z_v(g_v)$ has only one term $Z(1)_{K_v}$ which is an identity operator. For each v, $Z_v(g_v)$ is an operator which is étale outside v.

Let S be a finite subset of F including all places in Σ and places where U_v is not maximal. Then the Shimura curve X_U has good reduction outside S. Using the same argument as in our previous paper [Zh1] we can show the following identity:

$$Z_v(g_v)\widehat{\xi_\omega} = Z_v(g,\omega)\widehat{\xi_\omega} + D(g_v)$$

where $D(g_v)$ is a constant multiple of the divisor ωX_v supported on the special fiber of $\mathscr{X}$ over v whose connected components is also indexed by $F_+^\times\backslash\mathbb{A}_f^\times$. The composition of two operator at places $u \neq v$ outside S is given as

$$Z_u(g_u)Z_v(g_v)\widehat{\xi_\omega} - Z_u(g_u,\omega_u)Z_v(g_v,\omega_v)\widehat{\xi_\omega} = Z_u(g_u,\omega_u)D(g_v) + Z_v(g_v,\omega_v)D(g_u).$$

We have shown that

$$g \longrightarrow D(g) := Z^S(g)\widehat{\xi_\omega} - Z^S(g,\omega^S)\widehat{\xi_\omega}$$

behaves like derivative for $g \in \mathrm{GL}_2(\mathbb{A}^S)$: for any two coprime $g_1, g_2 \in \mathrm{GL}_2(\mathbb{A}^S)$ in the sense that for all $v \notin S$, either g_1 or g_2 is in $\mathrm{GL}_2(\mathscr{O}_v)$ one has

$$D(g_1g_2) = Z^S(g_1,\omega^S)D(g_2) + Z^S(g_2,\omega^S)D(g_1).$$

Thus $D(g)$ is a derivation for $Z^S(g,\omega^S)$ as defined in §4.4.4 in our previous paper [Zh1].

Let us apply this to the function

$$g \longrightarrow \langle Z_\phi(g)\widehat{\xi}, \widehat{\eta}\rangle.$$

This function may not be automorphic but invariant under $P(F)$. Thus it has a Whittaker function

$$\langle Z_{\phi,\psi}(g)\widehat{\xi}, \widehat{\eta}\rangle.$$

Let us fix $g_S \in \mathrm{GL}_2(\mathbb{A}_S)$. Then we obtain a function on $g^S \in \mathrm{GL}_2(\mathbb{A}^S)$:

$$g^S \longrightarrow \langle Z_{\phi,\psi}(g_Sg^S)\omega\widehat{\xi}, \widehat{\eta}\rangle = \langle Z^S(g^S)\widehat{\xi}, Z_S^t(g_S)\widehat{\eta}\rangle$$

where $Z_S^t(g_S)$ is the transpose of $Z_S(g_S)$. We have just shown that this is a derivation function for the Whittaker function of the Eisenstein series of $E(g,\phi,\omega)$. Applying this to the decomposition of $Z_\phi(g,t_1,t_2)$ we obtain the following:

Proposition 4.6.1. *For fixed ϕ, there is an S such that for any t_1, t_2, the Whittaker function of the difference*

$$Z_\phi(g,t_1,t_2) - \langle Z_\phi(g)t_1, t_2\rangle$$

is a finite linear combination of the Whittaker functions of the Eisenstein series $E(g,\phi_i,\omega_i)$ and their derivations where ϕ_i are pure tensors and ω_i are finite idele class characters.

5 Local heights of CM points

In this section, we compute the local heights of CM-points

$$\langle Z_\phi^* t_1, t_2\rangle = i(Z_\phi^* t_1, t_2) + j(Z_\phi^* t_1, t_2)$$

and compare it with the local analytic kernel function $\mathscr{K}_\phi^{(v)}$ appeared in the decomposition of $I'(0,g)$ representing series $L'(1/2,\pi,\chi)$. We will follow the work of Gross–Zagier and its extension in our previous paper [Zh2].

In §5.1, we study the case of archimedean place. We obtain an explicit formula which agrees with the analytic kernel. In §5.2, we study the supersingular case, namely the finite place not in Σ and inert in E. In the unramified case, the formula is also explicit and agree with corresponding analytic kernels. In the ramified case, under the condition that ϕ_v is supported in certain non-degenerate locus of $\mathbb{B}_v \times F_v^\times$, we can show that the i-part of the local intersection is given by certain pseudo-theta series defined in the next section. The same result holds at a superspecial place v in §5.3 (namely, a finite place in Σ). In §5.4, we treat ordinary case, namely finite places split in E. Again the formula in the unramified case is explicit and agrees with analytic kernels. The i-part of the ramified case will contribute pseudo-theta series. In §5.5, we will show that the j-part of local intersection contribute pseudo-theta series for ϕ_v with non-generate supports.

5.1 Archimedean places

In this subsection we want to describe local heights of CM points at any archimedean place v. Fix an identification $B(\mathbb{A}_f) = \mathbb{B}_f$. We will use the uniformization

$$X_{U,v}(\mathbb{C}) = B_+^\times \backslash \mathscr{H} \times B^\times(\mathbb{A}_f)/U$$

where $B = B(v)$. We follow the treatment of Gross-Zagier [GZ]. See also [Zh2].

Basic formula

For any two points $z_1, z_2 \in \mathscr{H}$, the hyperbolic cosine of the hyperbolic distance between them is given by

$$d(z_1, z_2) = 1 + \frac{|z_1 - z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}.$$

It is invariant under the action of $\mathrm{GL}_2(\mathbb{R})$. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, denote

$$m_s(z_1, z_2) = Q_s(d(z_1, z_2)),$$

where

$$Q_s(t) = \int_0^\infty \left(t + \sqrt{t^2-1}\cosh u\right)^{-1-s} du$$

is the Legendre function of the second kind. Note that

$$Q_0(1+2\lambda) = \frac{1}{2}\log(1+\frac{1}{\lambda}), \quad \lambda > 0.$$

We see that $m_0(z_1, z_2)$ has the right logarithmic singularity.

For any two distinct points of

$$X_{U,v}(\mathbb{C}) = B_+^\times \backslash \mathscr{H} \times B^\times(\mathbb{A}_f)/U$$

represented by $(z_1, \beta_1), (z_2, \beta_2) \in \mathscr{H} \times B^\times(\mathbb{A}_f)$, we denote

$$g_s((z_1, \beta_1), (z_2, \beta_2)) = \sum_{\gamma \in \mu_U \backslash B_+^\times} m_s(z_1, \gamma z_2)\ 1_U(\beta_1^{-1}\gamma\beta_2).$$

It is easy to see that the sum is independent of the choice of the representatives $(z_1, \beta_1), (z_2, \beta_2)$, and hence defines a pairing on $X_{U,v}(\mathbb{C})$. Then the local height is given by

$$i_{\bar{v}}((z_1, \beta_1), (z_2, \beta_2)) = \widetilde{\lim}_{s\to 0}\ g_s((z_1, \beta_1), (z_2, \beta_2)).$$

Here $\widetilde{\lim}_{s\to 0}$ denotes the constant term at $s=0$ of $g_s((z_1, \beta_1), (z_2, \beta_2))$, which converges for $\mathrm{Re}(s) > 0$ and has meromorphic continuation to $s=0$ with a simple pole.

The definition above uses adelic language, but it is not hard to convert it to the classical language. We first observe that $g_s((z_1, \beta_1), (z_2, \beta_2)) \neq 0$ only if there is a $\gamma_0 \in B_+^\times$ such that $\beta_1 \in \gamma_0\beta_2 U$, which just means that $(z_1, \beta_1), (z_2, \beta_2)$ are in the same connected component. Assuming this, then $(z_2, \beta_2) = (z_2', \beta_1)$ where $z_2' = \gamma_0 z_2$. we have

$$g_s((z_1, \beta_1), (z_2, \beta_2)) = g_s((z_1, \beta_1), (z_2', \beta_1)) = \sum_{\gamma \in \mu_U \backslash B_+^\times} m_s(z_1, \gamma z_2')\ 1_U(\beta_1^{-1}\gamma\beta_1) = \sum_{\gamma \in \mu_U \backslash \Gamma} m_s(z_1, \gamma z_2').$$

Here we denote $\Gamma = B_+^\times \cap \beta_1 U \beta_1^{-1}$. The connected component of these two points is exactly

$$B_+^\times \backslash \mathscr{H} \times B_+^\times \beta_1 U/U \approx \Gamma \backslash \mathscr{H}, \qquad (z, b\beta_1 U) \mapsto b^{-1}z.$$

The stabilizer of $\mathscr{H}$ in Γ is exactly $\Gamma \cap F^\times = \mu_U$. Now we see that the formula is the same as those in [GZ] and [Zh2].

Next we consider the special case of CM points. For any $\gamma \in B_{v,+}^\times - E_v^\times$, we have

$$1 + \frac{|z_0 - \gamma z_0|^2}{2\mathrm{Im}(z_0)\mathrm{Im}(\gamma z_0)} = 1 - 2\xi(\gamma).$$

Thus it is convenient to denote

$$m_s(\gamma) = Q_s(1 - 2\xi(\gamma)), \qquad \gamma \in B_v^\times - E_v^\times.$$

For any two distinct CM points $\beta_1, \beta_2 \in \mathrm{CM}_U$, we obtain

$$g_s(\beta_1, \beta_2) = \sum_{\gamma \in \mu_U \backslash B_+^\times} m_s(\gamma)\ 1_U(\beta_1^{-1}\gamma\beta_2),$$

and

$$i_{\bar{v}}(\beta_1, \beta_2) = \widetilde{\lim}_{s \to 0} g_s(\beta_1, \beta_2).$$

Note that $m_s(\gamma)$ is not defined for $\gamma \in E^\times$. The summation above makes sense because $\beta_1 \neq \beta_2$ implies that $1_U(\beta_1^{-1}\gamma\beta_2) = 0$ for all $\gamma \in E^\times$. But this is not true for self-intersections. In general, for any $\beta_1, \beta_2 \in \mathrm{CM}_U$, we introduce formally

$$g_s(\beta_1, \beta_2) = \sum_{\gamma \in \mu_U \backslash (B_+^\times - E^\times)} m_s(\gamma)\ 1_U(\beta_1^{-1}\gamma\beta_2),$$

and

$$i_{\bar{v}}(\beta_1, \beta_2) = \widetilde{\lim}_{s \to 0} g_s(\beta_1, \beta_2).$$

Then this definition is the same as the before if $\beta_1 \neq \beta_2$. In the case that $\beta_1 = \beta_2$, it will occur from the arithmetic adjunction formula.

Kernel function

In this section we compute the local height

$$i_{\bar{v}}(Z_\phi^*(g)t_1, t_2) \ = \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a i_{\bar{v}}(t_1 x, t_2).$$

for any archimedean place v of F.

The goal is to show that it is equal to $\widetilde{\mathscr{K}}_\phi^{(v)}(g, (t_1, t_2))$ obtained in Proposition 3.7.1.

Proposition 5.1.1. *For any $t_1, t_2 \in C_U$,*

$$i_{\bar{v}}(Z_\phi^*(g)t_1, t_2) = \mathscr{M}_\phi^{(v)}(g, (t_1, t_2)) = \sum_{a \in F^\times} \widetilde{\lim}_{s \to 0} \sum_{y \in \mu_U \backslash (B_+^\times - E^\times)} r(g, (t_1, t_2))\phi(y)_a m_s(y).$$

Therefore, $\mathscr{M}_\phi^{(v)}(g, (t_1, t_2)) = \widetilde{\mathscr{K}}_\phi^{(v)}(g, (t_1, t_2))$.

Proof. By the above formula,

$$\begin{aligned} i_{\bar{v}}(Z_\phi^*(g)t_1, t_2) &= \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a\ \widetilde{\lim}_{s \to 0} \sum_{\gamma \in \mu_U \backslash (B_+^\times - E^\times)} m_s(\gamma)\ 1_U(x^{-1}t_1^{-1}\gamma t_2) \\ &= \sum_{a \in F^\times} \widetilde{\lim}_{s \to 0} \sum_{\gamma \in \mu_U \backslash (B_+^\times - E^\times)} r(g)\phi(t_1^{-1}\gamma t_2)_a m_s(\gamma) \\ &= \sum_{a \in F^\times} \widetilde{\lim}_{s \to 0} \sum_{\gamma \in \mu_U \backslash (B_+^\times - E^\times)} r(g, (t_1, t_2))\phi(\gamma)_a m_s(\gamma). \end{aligned}$$

Here the second equality is obtained by replacing x by $t_1^{-1}\gamma t_2$. □

We want to compare the above result with the holomorphic projection

$$\widetilde{\mathscr{K}_\phi^{(v)}}(g,(t_1,t_2)) = \sum_{a\in F^\times} \widetilde{\lim}_{s\to 0} \sum_{y\in\mu_U\backslash(B_+^\times - E^\times)} r(g,(t_1,t_2))\phi(y)_a\, k_{v,s}(y)$$

computed in Proposition 3.7.1.

It amounts to compare

$$m_s(y) = Q_s(1-2\xi(y))$$

with

$$k_{v,s}(y) = \frac{\Gamma(s+1)}{2(4\pi)^s}\int_1^\infty \frac{1}{t(1-\xi_v(y)t)^{s+1}}dt.$$

By the result of Gross-Zagier,

$$\int_1^\infty \frac{1}{t(1-\xi t)^{s+1}}dt = 2Q_s(1-2\xi) + O(|\xi|^{-s-2}), \quad \xi\to-\infty,$$

and the error term vanishes at $s=0$. We conclude that

$$\widetilde{\mathscr{K}_\phi^{(v)}}(g,(t_1,t_2)) = i_{\bar v}(Z_\phi^*(g)t_1, t_2).$$

5.2 Supersingular case

Let v be a finite prime of F non-split in E but split in $\mathbb{B}$. We consider the local height at the unique place of E lying over v. We will use the local multiplicity functions treated in Zhang [Zh2]. For more details, we refer to that paper.

Multiplicity function

Assume that U is decomposable at v:

$$U = U_v\cdot U^v.$$

Let $B = B(v)$ be the nearby quaternion algebra over F. Make an identification $B(\mathbb{A}^v) = \mathbb{B}^v$. Then the set of supersingular points on X_K over v is parameterized by

$$\mathrm{SS}_U = B^\times\backslash(F_v^\times/\det(U_v))\times(\mathbb{B}_f^{v\times}/U^v).$$

Then we have a natural isomorphism

$$\bar v: \mathrm{CM}_U = E^\times\backslash\mathbb{B}_f^\times/U \to B^\times\backslash(B^\times\times_{E^\times}\mathbb{B}_v^\times/U_v)\times\mathbb{B}_f^{v\times}/U^v$$

sending β to $(1,\beta_v,\beta^v)$. The reduction map $\mathrm{CM}_U\to\mathrm{SS}_U$ is given by taking norm on the first factor:

$$q:\quad B^\times\times_{E^\times}\mathbb{B}_v^\times\to F_v^\times,\qquad (b,\beta)\mapsto q(b)q(\beta).$$

The intersection pairing is given by a multiplicity function m on

$$\mathscr{H}_v := B_v^\times \times_{E_v^\times} \mathbb{B}_v^\times / U_v.$$

More precisely, the intersection of two points $(b_1, \beta_1), (b_2, \beta_2) \in \mathscr{H}_v$ is given by

$$g_v((b_1, \beta_1), (b_2, \beta_2)) = m(b_1^{-1} b_2, \beta_1^{-1} \beta_2).$$

The multiplicity function m is defined everywhere in $\mathscr{H}_v$ except at the image of $(1,1)$. It satisfies the property

$$m(b, \beta) = m(b^{-1}, \beta^{-1}).$$

Lemma 5.2.1. *For any two distinct CM-points $\beta_1 \in \mathrm{CM}_U$ and $t_2 \in C_U$, their local height is given by*

$$i_{\bar{v}}(\beta_1, t_2) = \sum_{\gamma \in \mu_U \backslash B^\times} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1} \gamma t_2^v).$$

Proof. Like the archimedean case, we compute the height by pulling back to $\mathscr{H}_v \times \mathbb{B}_f^{v\times}$. The height is the sum over $\gamma \in \mu_U \backslash B^\times$ of the intersection of $(1, \beta_{1v}, \beta_1^v)$ with $\gamma(1, t_{2v}, t_2^v) = (\gamma, t_{2v}, \gamma t_2^v) = (\gamma t_{2v}, 1, \gamma t_2^v)$ on $\mathscr{H}_v \times \mathbb{B}_f^{v\times}$. □

Analogous to the archimedean case, the summation is well-defined for all $\beta_1 \neq t_2$. But if $\beta_1 = t_2$, then the summation is not defined at $\gamma \in E^\times$ such that $\beta_1^{-1} \gamma t_2 \in U$. Hence, we extend the definition as

$$\begin{aligned}
& i_{\bar{v}}(\beta_1, t_2) \\
= & \sum_{\gamma \in \mu_U \backslash (B^\times - E^\times \cap \beta_1 U t_2^{-1})} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1} \gamma t_2^v) \\
= & \sum_{\gamma \in \mu_U \backslash (B^\times - E^\times)} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1} \gamma t_2^v) + \sum_{\gamma \in \mu_U \backslash (E^\times - \beta_1 U t_2^{-1})} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1} \gamma t_2^v) \\
= & \sum_{\gamma \in \mu_U \backslash (B^\times - E^\times)} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1} \gamma t_2^v) + \sum_{\gamma \in \mu_U \backslash (E^\times - \beta_1 U_v t_2^{-1})} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1} \gamma t_2^v).
\end{aligned}$$

The definition is the equal to the previous one if $\beta_1 \neq t_2$.

The kernel function

Now we compute

$$i_{\bar{v}}(Z_\phi^*(g) t_1, t_2) = \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g) \phi(x)_a i_{\bar{v}}(t_1 x, t_2).$$

By the above formula,

$$\begin{aligned} i_{\bar v}(Z^*_\phi(g)t_1,t_2) &= \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a \sum_{\gamma\in\mu_U\backslash(B^\times-E^\times)} m(\gamma t_2, x^{-1}t_1^{-1})1_{U^v}(x^{-1}t_1^{-1}\gamma t_2) \\ &\quad + \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a \sum_{\gamma\in\mu_U\backslash(E^\times-t_1xU_vt_2^{-1})} m(\gamma t_2, x^{-1}t_1^{-1})1_{U^v}(x^{-1}t_1^{-1}\gamma t_2). \end{aligned}$$

In the first triple sum, we replace x^v by $t_1^{-1}\gamma t_2$ and thus get

$$\begin{aligned} &\sum_{a\in F^\times}\sum_{\gamma\in\mu_U\backslash(B^\times-E^\times)} r(g)\phi^v(t_1^{-1}\gamma t_2)_a \sum_{x_v\in\mathbb{B}_v^\times/U_v} r(g)\phi_v(x_v)_a m(t_1^{-1}\gamma t_2, x^{-1}) \\ = &\sum_{u\in\mu_U^2\backslash F^\times}\sum_{\gamma\in B-E} r(g,(t_1,t_2))\phi^v(\gamma,u) \sum_{x_v\in\mathbb{B}_v^\times/U_v} r(g)\phi_v(x_v, uq(\gamma)/q(x_v))m(t_1^{-1}\gamma t_2, x^{-1}). \end{aligned}$$

In the second triple sum, we still replace x^v by $t_1^{-1}\gamma t_2$. The condition $\gamma\notin t_1xU_vt_2^{-1}$ is equivalent to $x_v\notin t_1^{-1}\gamma t_2U_v$. We get

$$\begin{aligned} &\sum_{a\in F^\times}\sum_{\gamma\in\mu_U\backslash E^\times} r(g)\phi^v(t_1^{-1}\gamma t_2)_a \sum_{x_v\in(\mathbb{B}_v^\times-t_1^{-1}\gamma t_2U_v)/U_v} r(g)\phi_v(x_v)_a m(t_1^{-1}\gamma t_2, x^{-1}) \\ = &\sum_{u\in\mu_U^2\backslash F^\times}\sum_{\gamma\in E^\times} r(g,(t_1,t_2))\phi^v(\gamma,u) \sum_{x_v\in(\mathbb{B}_v^\times-t_1^{-1}\gamma t_2U_v)/U_v} r(g)\phi_v(x_v, uq(\gamma)/q(x_v))m(t_1^{-1}\gamma t_2, x^{-1}). \end{aligned}$$

For convenience, we introduce:

Notation.

$$\begin{aligned} m_{\phi_v}(y,u) &= \int_{\mathbb{B}_v^\times} m(y,x^{-1})\phi_v(x,uq(y)/q(x))dx \\ &= \int_{\mathbb{B}_v^\times} m(y^{-1},x)\phi_v(x,uq(y)/q(x))dx, \quad (y,u)\in(B_v-E_v)\times F_v^\times \\ n_{\phi_v}(y,u) &= \int_{\mathbb{B}_v^\times-yU_v} m(y,x^{-1})\phi_v(x,uq(y)/q(x))dx \\ &= \int_{\mathbb{B}_v^\times-U_v} m(1,x)\phi_v(yx,u/q(x))dx, \quad (y,u)\in E_v^\times\times F_v^\times \end{aligned}$$

By this notation, we obtain:

Proposition 5.2.2.

$$i_{\bar v}(Z^*_\phi(g)t_1,t_2) = \mathscr{M}_\phi^{(v)}(g,(t_1,t_2)) + \mathscr{N}_\phi^{(v)}(g,(t_1,t_2))$$

where

$$\begin{aligned} \mathscr{M}_\phi^{(v)}(g,(t_1,t_2)) &= \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B-E} r(g,(t_1,t_2))\phi^v(y,u)\ m_{r(g,(t_1,t_2))\phi_v}(y,u), \\ \mathscr{N}_\phi^{(v)}(g,(t_1,t_2)) &= \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} r(g,(t_1,t_2))\phi^v(y,u)\ r(t_1,t_2)n_{r(g)\phi_v}(y,u). \end{aligned}$$

Here we use the convention

$$r(t_1,t_2)n_{r(g)\phi_v}(y,u) = n_{r(g)\phi_v}(t_1^{-1}yt_2, q(t_1t_2^{-1})u).$$

Note that in the above series, we write the dependence on (t_1,t_2) in different manners for m_{ϕ_v} and n_{ϕ_v}. This is because $m_{\phi_v}(y,u)$ translates well under the action of $P(F_v)\times(E_v^\times\times E_v^\times)$, but $n_{\phi_v}(y,u)$ only translates well under the action of $P(F_v)$. We should compare the following result with Lemma 3.2.2 for $k_{\phi_v}(g,y,u)$.

Lemma 5.2.3. *(1) The function $m_{r(g,(t_1,t_2))\phi_v}(y,u)$ behaves like Weil representation under the action of $P(F_v)\times(E_v^\times\times E_v^\times)$ on (y,u). Namely,*

$$m_{r(g,(t_1,t_2))\phi_v}(y,u) = r(g,(t_1,t_2))m_{\phi_v}(y,u), \quad (g,(t_1,t_2))\in P(F_v)\times(E_v^\times\times E_v^\times).$$

More precisely, it means that

$$\begin{aligned} m_{r(m(a))\phi_v}(y,u) &= |a|^2 m_{\phi_v}(ay,u), \quad a\in F_v^\times \\ m_{r(n(b))\phi_v}(y,u) &= \psi(buq(y))m_{\phi_v}(y,u), \quad b\in F_v \\ m_{r(d(c))\phi_v}(y,u) &= |c|^{-1}m_{\phi_v}(y,c^{-1}u), \quad c\in F_v^\times \\ m_{r(t_1,t_2)\phi_v}(g,y,u) &= m_{\phi_v}(t_1^{-1}yt_2, q(t_1t_2^{-1})u), \quad (t_1,t_2)\in E_v^\times\times E_v^\times \end{aligned}$$

(2) The function $n_{r(g)\phi_v}(y,u)$ behaves like Weil representation under the action of $P(F_v)$ on (y,u), i.e.,

$$\begin{aligned} n_{r(m(a))\phi_v}(y,u) &= |a|^2 n_{\phi_v}(ay,u), \quad a\in F_v^\times \\ n_{r(n(b))\phi_v}(y,u) &= \psi(buq(y))n_{\phi_v}(y,u), \quad b\in F_v \\ n_{r(d(c))\phi_v}(y,u) &= |c|^{-1}n_{\phi_v}(y,c^{-1}u), \quad c\in F_v^\times. \end{aligned}$$

Proof. They follow from basic properties of the multiplicity function $m(x,y)$. We only verify the first identity.

$$\begin{aligned} m_{r(m(a))\phi_v}(y,u) &= \int_{\mathrm{GL}_2(F_v)} m(y^{-1},x)r(g_v)\phi_v(ax,uq(y)/q(x))|a|^2dx \\ &= |a|^2\int_{\mathrm{GL}_2(F_v)} m(y^{-1},a^{-1}x)r(g_v)\phi_v(x,uq(y)/q(a^{-1}x))dx \\ &= |a|^2\int_{\mathrm{GL}_2(F_v)} m((ay)^{-1},x)r(g_v)\phi_v(x,uq(ay)/q(x))dx \\ &= |a|^2 m_{\phi_v}(ay,u). \end{aligned}$$

□

Unramified Case

Fixing an isomorphism $\mathbb{B}_v = M_2(F_v)$. In this subsection we compute $m_{\phi_v}(y,u)$ and $n_{\phi_v}(y,u)$ in the following unramified case:

1. ϕ_v is the characteristic function of $M_2(O_{F_v}) \times O_{F_v}^\times$;
2. U_v is the maximal compact subgroup $\mathrm{GL}_2(O_{F_v})$.

By [Zh2], there is a decomposition

$$\mathrm{GL}_2(F_v) = \coprod_{c=0}^{\infty} E_v^\times h_c \mathrm{GL}_2(O_{F_v}), \qquad h_c = \begin{pmatrix} 1 & 0 \\ 0 & \varpi^c \end{pmatrix} \tag{5.2.1}$$

We may assume that $\beta = h_c$. For any $b \in B_v$ we set

$$\xi(b) = \frac{q(b)}{q(b)} = \frac{q(b_2)}{q(b)}$$

for the orthogonal decomposition $b = b_1 + b_2$. The following result is Lemma 5.5.2 in [Zh2]. There is a small mistake in the original statement. Here is the corrected one.

Lemma 5.2.4. *The multiplicity function $m(b,\beta) \neq 0$ only if $q(b)q(\beta) \in O_{F_v}^\times$. In this case, assume that $\beta \in E_v^\times h_c \mathrm{GL}_2(O_{F_v})$. Then*

$$m(b,\beta) = \begin{cases} \frac{1}{2}(\mathrm{ord}_v \xi(b) + 1) & \text{if } c = 0; \\ N_v^{1-c}(N_v+1)^{-1} & \text{if } c > 0,\ E_v/F_v \text{ is unramified;} \\ \frac{1}{2} N_v^{-c} & \text{if } c > 0,\ E_v/F_v \text{ is ramified.} \end{cases}$$

Proposition 5.2.5. *(1) The function $m_{\phi_v}(y,u) \neq 0$ only if $(y,u) \in O_{B_v} \times O_{F_v}^\times$. In this case,*

$$m_{\phi_v}(y,u) = \frac{1}{2}(\mathrm{ord}_v q(y_2) + 1).$$

(2) The function $n_{\phi_v}(y,u) \neq 0$ only if $(y,u) \in O_{E_v} \times O_{F_v}^\times$. In this case,

$$n_{\phi_v}(y,u) = \frac{1}{2}\mathrm{ord}_v q(y).$$

Proof. We will use Lemma 5.2.4. Recall that

$$m_{\phi_v}(y,u) = \sum_{x \in \mathrm{GL}_2(F_v)/U_v} m(y^{-1}, x)\phi_v(x, uq(y)/q(x)).$$

Note that $m(y^{-1},x) \neq 0$ only if $\mathrm{ord}(q(x)/q(y)) = 0$. Under this condition, $\phi_v(x, uq(y)/q(x)) \neq 0$ if and only if $u \in O_{F_v}^\times$ and $x \in M_2(O_{F_v})$. It follows that $m_{\phi_v}(y,u) \neq 0$ only if $u \in O_{F_v}^\times$ and $n = \mathrm{ord}(q(y)) \geq 0$. Assuming these two conditions, we have

$$m_{\phi_v}(y,u) = \sum_{x \in M_2(O_{F_v})_n/U_v} m(y^{-1}, x),$$

where $M_2(O_{F_v})_n$ denotes the set of integral matrices whose determinants have valuation n. Now we use decomposition (2.3), we obtain

$$m_{\phi_v}(y,u) = \sum_{c=0}^{\infty} m(y^{-1}, h_c)\mathrm{vol}(E_v^\times h_c \mathrm{GL}_2(O_{F_v}) \cap M_2(O_{F_v})_n).$$

We first consider the case that E_v/F_v is unramified. The set in the right hand side is non-empty only if $n-c$ is even and non-negative. In this case it is given by

$$\varpi^{(n-c)/2} O_{E_v}^\times h_c U_v.$$

The volume of this set is 1 if $c=0$ and $N_v^{c-1}(N_v+1)$ if $c>0$ by the computation of [Zh2, p. 101]. It follows that, for $c>0$ with $2 \mid (n-c)$,

$$m(y^{-1}, h_c)\mathrm{vol}(E_v^\times h_c \mathrm{GL}_2(O_{F_v}) \cap M_2(O_{F_v})_n) = 1.$$

If n is even,

$$m_{\phi_v}(y,u) = \frac{1}{2}(\mathrm{ord}_v \xi(y)+1) + \frac{n}{2} = \frac{1}{2}(\mathrm{ord}_v q(y_2)+1).$$

If n is odd, $m_{\phi_v}(y,u) = \dfrac{1}{2}(n+1)$. It is easy to see that $\mathrm{ord}_v q(y_1)$ is even and $\mathrm{ord}_v q(y_2)$ is odd. Then $n = \mathrm{ord}_v q(y_2)$, since $n = \mathrm{ord}_v q(y) = \min\{\mathrm{ord}_v q(y_1), \mathrm{ord}_v q(y_2)\}$ is odd. We still get $m_{\phi_v}(y,u) = \frac{1}{2}(\mathrm{ord}_v q(y_2)+1)$ in this case.

Now assume that E_v/F_v is ramified. Then the condition that $2 \mid (n-c)$ is unnecessary, and $\mathrm{vol}(E_v^\times h_c \mathrm{GL}_2(O_{F_v}) \cap M_2(O_{F_v})_n) = N_v^c$. Thus

$$m_{\phi_v}(y,u) = \frac{1}{2}(\mathrm{ord}_v \xi(y)+1) + n \cdot \frac{1}{2} = \frac{1}{2}(\mathrm{ord}_v q(y_2)+1).$$

As for n_{ϕ_v}, still write $n = \mathrm{ord}_v q(y)$, and we have

$$n_{\phi_v}(y,u) = \sum_{c=1}^{\infty} m(y^{-1}, h_c)\mathrm{vol}(E_v^\times h_c \mathrm{GL}_2(O_{F_v}) \cap M_2(O_{F_v})_n).$$

If E_v/F_v is unramified, n is always even, and we have $m_{\phi_v}(y,u) = \dfrac{n}{2}$. The ramified case is similar. $\square$

We immediately see that in the unramified case, m_{ϕ_v} matches the analytic kernel k_{ϕ_v} computed in Proposition 3.4.2. As for $n_{\phi_v}(y,u)$, its counterpart coming from $\mathscr{P}rI'(0,g)$ is exactly $\log \delta(g_v) + \dfrac{1}{2}\log |uq(y)|_v$ in Proposition 3.7.1. The good news is that the extra term $\log \delta(g_v)$ cancels the action of g. In summary, we have the following result:

Proposition 5.2.6. *(1) Let v be unramified as above. Then*

$$n_{r(g)\phi_v}(y,u)\log N_v = -(\log \delta(g_v) + \frac{1}{2}\log |uq(y)|_v)\ r(g)\phi_v(y,u).$$

(2) Let v be unramified as above with further conditions:

- *$v \nmid 2$ if E_v/F_v is ramified;*
- *the local different d_v is trivial.*

Then

$$k_{r(t_1,t_2)\phi_v}(g, y, u) = m_{r(g,(t_1,t_2))\phi_v}(y, u) \log N_v,$$

and thus

$$\mathscr{K}_\phi^{(v)}(g, (t_1, t_2)) = \mathscr{M}_\phi^{(v)}(g, (t_1, t_2)) \log N_v.$$

Proof. We first consider (2). The case $(g, t_1, t_2) = (1, 1, 1)$ follows from the above result and Corollary 3.4.2. It is also true for $g \in \mathrm{GL}_2(O_{F_v})$ since it is easy to see that such g will not change the local kernel functions for standard ϕ_v. For the general case, apply the action of $P(F_v)$ and $E_v^\times \times E_v^\times$. The equality follows from Proposition 3.2.2 and Lemma 5.2.3. Similarly, by Lemma 5.2.3 we can show (1). □

General case

Now we consider general U_v. By Proposition 5.2.5 for the unramified case, we know that m_{ϕ_v} may have logarithmic singularity around the boundary $E_v \times F_v^\times$, and n_{ϕ_v} may have logarithmic singularity around the boundary $\{0\} \times F_v^\times$. These singularities are caused by self-intersections in the computation of local multiplicity. However, we will see that there is no singularity if $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$.

Proposition 5.2.7. *Assume that $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ is invariant under the right action of U_v. Then:*

(1) The function $m_{\phi_v}(y, u)$ can be extended to a Schwartz function for $(y, u) \in B_v \times F_v^\times$.

(2) The function $n_{\phi_v}(y, u)$ can be extended to a Schwartz function for $(y, u) \in E_v \times F_v^\times$.

Proof. We only consider m_{ϕ_v}. The proof for n_{ϕ_v} is similar.

By the choice of ϕ_v, there is a constant $C > 0$ such that $\phi_v(x, u) \neq 0$ only if $-C < v(q(x)) < C$ and $-C < v(u) < C$. Recall that

$$m_{\phi_v}(y, u) = \int_{\mathbb{B}_v^\times} m(y^{-1}, x)\phi_v(x, uq(y)/q(x))dx.$$

In order that $m(y^{-1}, x) \neq 0$, we have to make $q(y)/q(x) \in q(U_v)$ and thus $v(q(y)) = v(q(x))$. It follows that $\phi_v(x, uq(y)/q(x)) \neq 0$ only if $-C < v(u) < C$. The same is true for $m_{\phi_v}(y, u)$ by looking at the integral above. Then it is easy to see that $m_{\phi_v}(y, u)$ is Schwartz for $u \in F_v^\times$.

On the other hand, $m_{\phi_v}(y, u) \neq 0$ only if $-C < v(q(y)) < C$, since $\phi_v(x, uq(y)/q(x)) \neq 0$ only if $-C < v(q(x)) < C$. Extend m_{ϕ_v} to $B_v \times F^\times$ by taking zero outside $B_v^\times \times F^\times$. We only need to show that it is locally constant in $B_v^\times \times F^\times$.

We have $\phi_v(E_vU_v, F_v^\times) = 0$ by the assumption that $\phi_v(E_v, F_v^\times) = 0$ and ϕ_v is invariant under U_v. Thus

$$m_{\phi_v}(y,u) = \int_{\mathbb{B}_v^\times} m(y^{-1},x)(1-1_{E_v^\times U_v}(x))\phi_v(x, uq(y)/q(x))dx.$$

It is locally constant in $B_v^\times \times F_v^\times$, since $m(y^{-1},x)(1-1_{E_v^\times U_v}(x))$ is locally constant as a function on $B_v^\times \times \mathbb{B}_v^\times$. This completes the proof. □

5.3 Superspecial case

Let v be a finite prime of F non-split in both B and E. In this section we consider the local height

$$i_{\bar{v}}(Z_\phi^*(g)t_1, t_2) = \sum_{a\in F^\times} \sum_{x\in \mathbb{B}_f^\times/U} r(g)\phi(x)_a i_{\bar{v}}(t_1x, t_2).$$

Since most of the computations are similar to the supersingular case, sometimes we will only list the results.

Denote by $B = B(v)$ the nearby quaternion algebra. We fix identifications $B_v \simeq M_2(F_v)$ and $B(\mathbb{A}_f^v) \simeq \mathbb{B}_f^v$. The intersection pairing is given by a multiplicity function m on

$$\mathscr{H}_v := B_v^\times \times_{E_v^\times} \mathbb{B}_v^\times/U_v.$$

More precisely, the intersection of two points $(b_1, \beta_1), (b_2, \beta_2) \in \mathscr{H}_v$ is given by

$$g_v((b_1,\beta_1),(b_2,\beta_2)) = m(b_1^{-1}b_2, \beta_1^{-1}\beta_2).$$

The multiplicity function m is defined everywhere on $\mathscr{H}_v$ except at the image of $(1,1)$. It satisfies the property

$$m(b,\beta) = m(b^{-1}, \beta^{-1}).$$

For any two distinct CM-points $\beta_1 \in \mathrm{CM}_U$ and $t_2 \in C_U$, their local height is given by

$$i_{\bar{v}}(\beta_1, t_2) = \sum_{\gamma\in\mu_U\backslash B^\times} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1}\gamma t_2^v).$$

If $\beta_1 = t_2$, then the summation is not defined at $\gamma \in E^\times$ such that $\beta_1^{-1}\gamma t_2 \in U$. So we extend the definition as

$$\begin{aligned} &i_{\bar{v}}(\beta_1, t_2) \\ =& \sum_{\gamma\in\mu_U\backslash(B^\times - E^\times\cap\beta_1Ut_2^{-1})} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1}\gamma t_2^v) \\ =& \sum_{\gamma\in\mu_U\backslash(B^\times - E^\times)} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1}\gamma t_2^v) + \sum_{\gamma\in\mu_U\backslash(E^\times - \beta_1U_vt_2^{-1})} m(\gamma t_{2v}, \beta_{1v}^{-1}) 1_{U^v}((\beta_1^v)^{-1}\gamma t_2^v). \end{aligned}$$

Analogous to Proposition 5.2.2, we have the following result.

Proposition 5.3.1.

$$i_{\bar{v}}(Z_\phi^*(g)t_1, t_2) = \mathscr{M}_\phi^{(v)}(g, (t_1, t_2)) + \mathscr{N}_\phi^{(v)}(g, (t_1, t_2))$$

where

$$\mathscr{M}_\phi^{(v)}(g, (t_1, t_2)) = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{y \in B - E} r(g, (t_1, t_2))\phi^v(y, u)\ m_{r(g,(t_1,t_2))\phi_v}(y, u),$$
$$\mathscr{N}_\phi^{(v)}(g, (t_1, t_2)) = \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{y \in E^\times} r(g, (t_1, t_2))\phi^v(y, u)\ r(t_1, t_2) n_{r(g)\phi_v}(y, u).$$

Here we use the same notations:

$$m_{\phi_v}(y, u) = \int_{\mathbb{B}_v^\times} m(y^{-1}, x)\phi_v(x, uq(y)/q(x))dx, \quad (y, u) \in (B_v - E_v) \times F_v^\times$$
$$n_{\phi_v}(y, u) = \int_{\mathbb{B}_v^\times - yU_v} m(y^{-1}, x)\phi_v(x, uq(y)/q(x))dx, \quad (y, u) \in E_v^\times \times F_v^\times$$

Lemma 5.2.3 is still true. It says that the action of $P(F_v) \times (E_v^\times \times E_v^\times)$ on $m_{r(g,(t_1,t_2))\phi_v}(y, u)$ and the action of $P(F_v)$ on $n_{r(g)\phi_v}(y, u)$ behave like Weil representation. The main result below is parallel to Proposition 5.2.7.

Proposition 5.3.2. *Assume $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ is invariant under the right action of U_v. Then:*

(1) The function $m_{\phi_v}(y, u)$ can be extended to a Schwartz function for $(y, u) \in B_v \times F_v^\times$.

(2) The function $n_{\phi_v}(y, u)$ can be extended to a Schwartz function for $(y, u) \in E_v \times F_v^\times$.

Proof. The proof is very similar to Proposition 5.2.7. We only sketch one for m_{ϕ_v}. The proof for n_{ϕ_v} is similar.

By the argument of Proposition 5.2.7, there is a constant $C > 0$ such that $m_{\phi_v}(y, u) \neq 0$ only if $-C < v(q(y)) < C$ and $-C < v(u) < C$. Extend m_{ϕ_v} to $B_v \times F^\times$ by taking zero outside $B_v^\times \times F^\times$. The same method shows that it is locally constant on $B_v^\times \times F^\times$. But this is not enough to conclude that it is compactly supported in y, since B_v is the matrix algebra this time. However, there is a compact subgroup D_v of $B_v^\times$ such that $m(y, x^{-1}) \neq 0$ only if $y \in E_v^\times D_v$. By this we can have our conclusion.

□

5.4 Ordinary case

Assume that v is a finite prime of F such that E_v is split. Then $\mathbb{B}_v$ is split because of the embedding $E_v \to \mathbb{B}_v$. Let v_1 and v_2 be the two primes of E lying over v. Fix an identification $\mathbb{B}_v \cong \mathrm{M}_2(F_v)$ under which $E_v = \begin{pmatrix} F_v & \\ & F_v \end{pmatrix}$. Assume that v_1 corresponds to the ideal $\begin{pmatrix} F_v & \\ & 0 \end{pmatrix}$ and v_2 corresponds to $\begin{pmatrix} 0 & \\ & F_v \end{pmatrix}$ of E_v.

We will make use of results of [Zh2]. The reduction map of CM-points to ordinary points at v_1 is given by

$$E^\times \backslash \mathbb{B}_f^\times / U \longrightarrow E^\times \backslash (N(F_v) \backslash \mathrm{GL}_2(F_v)) \times \mathbb{B}_f^{v\times}/U.$$

The intersection multiplicity is a function on $N(F_v)U_v/U_v$. By [Zh2, Lemma 6.3.2], the local pairing of $\beta_1, \beta_2 \in \mathrm{GL}_2(F_v)/U_v$ is given by

$$g_{v_1}(\beta_1,\beta_2) = \int_{\mathrm{GL}_2(F_v)} \int_{\mathrm{GL}_2(F_v)} 1_{\beta_1 U_v}(hy) 1_{\beta_2 U_v}(y) dy \tilde{d}h = \int_{\mathrm{GL}_2(F_v)} 1_{U_v}(\beta_1^{-1} h \beta_2) \tilde{d}h.$$

Here the measure dy is the usual Haar measure on $\mathrm{GL}_2(F_v)$, and the measure $\tilde{d}h$ is defined as follows:

$$\int_{\mathrm{GL}_2(F_v)} f(h) \tilde{d}h := \int_{F_v^\times} f(n(b)) d^\times b = \int_{F_v^\times} f \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} d^\times b.$$

A nice property is that $\tilde{d}h$ is invariant under the substitution $h \mapsto tht^{-1}$ for any $t \in E_v^\times$. And thus

$$\int_{\mathrm{GL}_2(F_v)} f(ht) \tilde{d}h = \int_{\mathrm{GL}_2(F_v)} f(th) \tilde{d}h.$$

By this pairing, the local height pairing of two distinct CM points $\beta_1, \beta_2 \in E^\times \backslash \mathbb{B}_f^\times / U$ is given by

$$i_{\bar{v}_1}(\beta_1,\beta_2) = \sum_{\gamma \in \mu_U \backslash E^\times} g_{v_1}(\beta_1, \gamma\beta_2) 1_{U^v}(\beta_1^{-1}\gamma\beta_2).$$

Just like the other cases, the above summation is only well-defined for $\beta_1 \neq \beta_2$. But we can extend the definition as

$$i_{\bar{v}_1}(\beta_1,\beta_2) = \sum_{\gamma \in \mu_U \backslash (E^\times - \beta_1 U \beta_2^{-1})} g_{v_1}(\beta_1, \gamma\beta_2) 1_{U^v}(\beta_1^{-1}\gamma\beta_2) = \sum_{\gamma \in \mu_U \backslash (E^\times - \beta_1 U_v \beta_2^{-1})} g_{v_1}(\beta_1, \gamma\beta_2) 1_{U^v}(\beta_1^{-1}\gamma\beta_2).$$

We want to compute

$$\begin{aligned} i_{\bar{v}_1}(Z_\phi^*(g)t_1, t_2) &= \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a \; i_{\bar{v}_1}(t_1 x, t_2) \\ &= \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a \sum_{\gamma \in \mu_U \backslash (E^\times - t_1 x U_v t_2^{-1})} g_{v_1}(t_1 x, \gamma t_2) 1_{U^v}(x^{-1} t_1^{-1} \gamma t_2). \end{aligned}$$

By $1_{U^v}(x^{-1}t_1^{-1}\gamma t_2) = 1$, we have $x^v \in t_1^{-1}\gamma t_2 U^v$; by $\gamma \notin t_1 x U_v t_2^{-1}$, we have $x_v \notin t_1^{-1}\gamma t_2 U_v$. Thus

$$i_{\bar{v}_1}(Z_\phi^*(g)t_1, t_2) = \sum_{a\in F^\times}\sum_{\gamma\in\mu_U\backslash E^\times} r(g)\phi^v(t_1^{-1}\gamma t_2)_a \sum_{x_v\in(\mathbb{B}_v^\times - t_1^{-1}\gamma t_2 U_v)/U_v} r(g)\phi_v(x_v)_a g_{v_1}(t_1 x_v, \gamma t_2).$$

Recall that

$$g_{v_1}(b_1, b_2) = \int_{\mathrm{GL}_2(F_v)} 1_{U_v}(b_1^{-1} h b_2)\tilde{d}h.$$

The last summation is equal to

$$\begin{aligned}
&\sum_{x_v\in(\mathbb{B}_v^\times - t_1^{-1}\gamma t_2 U_v)/U_v} r(g)\phi_v(x_v)_a \int_{\mathrm{GL}_2(F_v)} 1_{U_v}(x_v^{-1}t_1^{-1}h\gamma t_2)\tilde{d}h \\
=&\int_{\mathrm{GL}_2(F_v)} \sum_{x_v\in(\mathbb{B}_v^\times - t_1^{-1}\gamma t_2 U_v)/U_v} r(g)\phi_v(x_v)_a 1_{U_v}(x_v^{-1}t_1^{-1}\gamma t_2 h)\tilde{d}h \\
=&\int_{\mathrm{GL}_2(F_v)-U_v} r(g)\phi_v(t_1^{-1}\gamma t_2 h)_a \tilde{d}h.
\end{aligned}$$

Here we explain the last equality above. We have $x_v \in t_1^{-1}\gamma t_2 h U_v$ by $1_{U_v}(x_v^{-1} h t_1^{-1}\gamma t_2) = 1$. Then $h \notin U_v$ by $x_v \notin t_1^{-1}\gamma t_2 U_v$.

In summary, we have obtained:

Proposition 5.4.1.

$$i_{\bar{v}_1}(Z_\phi^*(g)t_1, t_2) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} r(g,(t_1,t_2))\phi^v(y,u)\ r(t_1,t_2) n_{r(g)\phi_v, v_1}(y,u).$$

where

$$\begin{aligned}
n_{\phi_v,v_1}(y,u) =& \int_{\mathrm{GL}_2(F_v)-U_v} \phi_v(yh,u)\tilde{d}h \\
=&\phi_{1,v}(y,u)\int_{F_v^\times} \phi_{2,v}\left(y\begin{pmatrix} 0 & b \\ & 0\end{pmatrix}, u\right)\left(1 - 1_{U_v}\left(\begin{pmatrix} 1 & b \\ & 1\end{pmatrix}\right)\right) d^\times b.
\end{aligned}$$

for any $(y,u) \in E_v^\times \times F_v^\times$.

The above result is only for the place v_1. By symmetry, we have a formula for v_2 after replacing the upper triangular matrices by lower triangular matrices. The average of these two results yields:

Proposition 5.4.2.

$$i_{\bar{v}}(Z_\phi^*(g)t_1, t_2) = \mathscr{N}_\phi^{(v)}(g,(t_1,t_2)) = \sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in E^\times} r(g,(t_1,t_2))\phi^v(y,u)\ r(t_1,t_2) n_{r(g)\phi_v}(y,u)$$

where

$$\begin{aligned} n_{\phi_v}(y,u) =& \frac{1}{2}\phi_{1,v}(y,u)\int_{F_v^\times}\phi_{2,v}\left(y\begin{pmatrix} 0 & b \\ & 0\end{pmatrix}, u\right)\left(1-1_{U_v}\left(\begin{pmatrix} 1 & b \\ & 1\end{pmatrix}\right)\right)d^\times b \\ &+\frac{1}{2}\phi_{1,v}(y,u)\int_{F_v^\times}\phi_{2,v}\left(y\begin{pmatrix} 0 & \\ b & 0\end{pmatrix}, u\right)\left(1-1_{U_v}\left(\begin{pmatrix} 1 & \\ b & 1\end{pmatrix}\right)\right)d^\times b \end{aligned}$$

for any $(y,u)\in E_v^\times\times F_v^\times$.

In the unramified case, we have

Proposition 5.4.3. *Assume that* U_v *is maximal and* ϕ_v *is the standard characteristic function. Then* $n_{\phi_v}(y,u)\neq 0$ *only if* $(y,u)\in O_{E_v}\times O_{F_v}^\times$. *In this case,*

$$n_{\phi_v}(y,u)=\frac{1}{2}\mathrm{ord}_v q(y).$$

Proof. Assume that $(y,u)\in O_{E_v}\times O_{F_v}^\times$. Otherwise, both sides are zero. Assume that $y=\begin{pmatrix} a_1 & \\ & a_2\end{pmatrix}$ under the identification.

Since U_v is maximal, $\begin{pmatrix} 1 & b \\ & 1\end{pmatrix}\in U_v$ if and only if $b\in O_{F_v}$. Then

$$n_{\phi_v,v_1}(y,u)=\int_{F_v-O_{F_v}}\phi_{2,v}\left(\begin{pmatrix} 0 & ba_1 \\ & 0\end{pmatrix}, u\right)d^\times b=\mathrm{ord}_v(a_1).$$

Similarly, the second integral is equal to $\mathrm{ord}_v(a_2)$. □

The above result looks exactly the same as the result of $n_{\phi_v}(y,u)$ in Proposition 5.2.5 in the supersingular case. This is the reason that we use the same notation for them. For example, we have the counterpart of Lemma 5.2.3 (2) as follows:

Lemma 5.4.4. *The function* $n_{r(g)\phi_v}(y,u)$ *behaves like Weil representation under the action of* $P(F_v)$ *on* (y,u), *i.e.,*

$$\begin{aligned} n_{r(m(a))\phi_v}(y,u) &= |a|^2 n_{\phi_v}(ay,u), \quad a\in F_v^\times \\ n_{r(n(b))\phi_v}(y,u) &= \psi(buq(y))n_{\phi_v}(y,u), \quad b\in F_v \\ n_{r(d(c))\phi_v}(y,u) &= |c|^{-1}n_{\phi_v}(y,c^{-1}u), \quad c\in F_v^\times. \end{aligned}$$

Analogous to Proposition 5.2.6, we have:

Proposition 5.4.5. *For the maximal* U_v *and the standard* ϕ_v *as in Proposition 5.4.3, we have*

$$n_{r(g)\phi_v}(y,u)=-(\log\delta(g_v)+\frac{1}{2}\log|uq(y)|_v)\ r(g)\phi_v(y,u).$$

Proof. We have the identity for $g = 1$ by Proposition 5.4.5. To extend the result to general $g \in \mathrm{GL}_2(F_v)$, we only need to apply Lemma 5.4.4 to the Iwasawa decomposition of g. □

Parallel to Proposition 5.2.7, we have

Proposition 5.4.6. *For general U_v, the function $n_{\phi_v}(y, u)$ can be extended to a Schwartz function for $(y, u) \in E_v \times F_v^\times$ if $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$.*

Proof. It suffices to show the result for

$$n_{\phi_v, v_1}(y, u) = \int_{F_v^\times} \phi_v \left(\left(\begin{array}{cc} a_1 & a_1 b \\ & a_2 \end{array} \right), u \right) \left(1 - 1_{U_v} \left(\left(\begin{array}{cc} 1 & b \\ & 1 \end{array} \right) \right) \right) d^\times b.$$

Here we still write $y = \left(\begin{array}{cc} a_1 & \\ & a_2 \end{array} \right)$.

It is easy to see that n_{ϕ_v, v_1} is locally constant at any (y, u) with $a_1 \neq 0$. The problem is to show that, for fixed $(y_0, u_0) \in F_v \times F_v^\times$, the function n_{ϕ_v, v_1} extends to a locally constant function at $(\tilde{y}_0, u_0)$. Here $\tilde{y}_0$ denotes the element $\left(\begin{array}{cc} 0 & \\ & y_0 \end{array} \right) \in E_v$.

Consider the behavior when (y, u) is closed to $(\tilde{y}_0, u_0)$. Since $v(a_1)$ is large, we see that $a_1 b$ is closed to zero when $\left(\begin{array}{cc} 1 & b \\ & 1 \end{array} \right) \in U_v$, and thus $\phi_v \left(\left(\begin{array}{cc} a_1 & a_1 b \\ & a_2 \end{array} \right), u \right) = \phi_v(\tilde{y}_0, u_0) = 0$. It follows that

$$n_{\phi_v, v_1}(y, u) = \int_{F_v^\times} \phi_v \left(\left(\begin{array}{cc} a_1 & a_1 b \\ & a_2 \end{array} \right), u \right) d^\times b = \int_{F_v^\times} \phi_v \left(\left(\begin{array}{cc} a_1 & b \\ & a_2 \end{array} \right), u \right) d^\times b.$$

It is independent of (y, u) in a neighborhood of $(\tilde{y}_0, u_0)$, and we can extend the function to $(\tilde{y}_0, u_0)$.

□

5.5 The j-part

Now we consider the pairing $j_{\bar{v}}(Z_\phi^*(g) t_1, t_2)$. It is nonzero only if v is a finite prime such that U_v is not maximal or B_v is nonsplit. The goal is to write it as a pseudo-theta series and control the singularities. It turns out that it has no singularity if $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$. In that case, an integration of the associated theta series becomes an Eisenstein series.

For any $\beta_1, \beta_2 \in \mathrm{CM}_U$, the pairing $j(\beta_1, \beta_2) \neq 0$ only if β_1, β_2 are in the same geometrically connected components of X_U. Once this is true, the pairing is given by a symmetric locally constant function $l(\cdot, \cdot)$ on $(\mathbb{B}_v^\times / U_v)^2$. In summary, we have

$$j_{\bar{v}}(\beta_1, \beta_2) = l(\beta_{1v}, \beta_{2v}) 1_{F_+^\times q(\beta_2) q(U)}(q(\beta_1)) = l(\beta_{1v}, \beta_{2v}) 1_{(\mathbb{B}_f)_{\mathrm{ad}} U}(\beta_2^{-1} \beta_1).$$

Here we denote

$$\begin{aligned}(\mathbb{B}_f)_{\mathrm{ad}} =& \{x \in \mathbb{B}_f : q(x) \in F_+^\times\}\\ (\mathbb{B}_f)_a =& \{x \in \mathbb{B}_f : q(x) = a\}, \quad a \in \mathbb{A}_f^\times.\end{aligned}$$

Now we go back to

$$\begin{aligned}& j_{\bar{v}}(Z^*_\phi(g)t_1, t_2)\\ =& \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a \ j_{\bar{v}}(t_1 x, t_2)\\ =& \sum_{a \in F^\times} \sum_{x \in \mathbb{B}_f^\times / U} r(g)\phi(x)_a \ l(t_1 x_v, t_2) 1_{(\mathbb{B}_f)_{\mathrm{ad}} U}(t_2^{-1} t_1 x)\\ =& \sum_{a \in F^\times} \sum_{x \in (\mathbb{B}_f)_{\mathrm{ad}} U/U} r(g)\phi(t_1^{-1} t_2 x)_a \ l(t_2 x_v, t_2)\\ =& \sum_{a \in F^\times} \sum_{x \in (\mathbb{B}_f)_{\mathrm{ad}}/U_{\mathrm{ad}}} r(g, (t_1 t_2^{-1}, 1))\phi(x)_a \ l(t_2 x_v, t_2).\end{aligned}$$

Apparently $\mu_U U^1 \subset U_{\mathrm{ad}}$. The quotient $U_{\mathrm{ad}}/\mu_U U^1 = (q(U) \cap F_+^\times)/\mu_U^2$ is finite since it is a quotient of two finitely generated abelian groups of the same rank. Hence we have

$$\begin{aligned}& [q(U) \cap F_+^\times : \mu_U^2] \ j_{\bar{v}}(Z^*_\phi(g)t_1, t_2)\\ =& \sum_{a \in F^\times} \sum_{x \in (\mathbb{B}_f)_{\mathrm{ad}}/\mu_U U^1} r(g, (t_1 t_2^{-1}, 1))\phi(x)_a \ l(t_2 x_v, t_2)\\ =& \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{x \in (\mathbb{B}_f)_{\mathrm{ad}}/U^1} r(g, (t_1 t_2^{-1}, 1))\phi(x, u) \ l(t_2 x_v, t_2)\\ =& \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{a \in F_+^\times} \sum_{x \in (\mathbb{B}_f)_a/U^1} r(g, (t_1 t_2^{-1}, 1))\phi(x, u) \ l(t_2 x_v, t_2)\\ =& \frac{1}{\mathrm{vol}(U^1)} \sum_{u \in \mu_U^2 \backslash F^\times} \sum_{a \in F_+^\times} \int_{(\mathbb{B}_f)_a} r(g, (t_1 t_2^{-1}, 1))\phi(x, u) \ l(t_2 x_v, t_2) dx.\end{aligned}$$

The integral above is a local product, which is nice at every place except v.

We first consider the case that $\mathbb{B}_v$ is a matrix algebra. We introduce a function

$$l_{\phi_v}(t_2, y, u) = \frac{1}{\mathrm{vol}(B_v^1)} \int_{(\mathbb{B}_v)_{q(y)}} \phi_v(x, u) \ l(t_2 x, t_2) dx, \quad (y, u) \in B_v^\times \times F_v^\times.$$

Note that B_v is a division algebra and thus B_v^1 is compact and has finite volume.

Fix $t_2 \in T(\mathbb{A}_f)$. In general, $l_{\phi_v}(t_2, y, u)$ is not defined at $y = 0$. But if $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$, then $\phi_v(x, F_v^\times) = 0$ when $v(q(x))$ is large. It follows that $l_{\phi_v}(t_2, y, u) = 0$ for y closed to 0. It is automatically a Schwartz function for $(y, u) \in B_v \times F_v^\times$.

Now we go back to the computation. We have:

$$
\begin{aligned}
&[q(U)\cap F_+^\times:\mu_U^2]\mathrm{vol}(U^1)\ j_{\bar v}(Z_\phi^*(g)t_1,t_2)\\
=&\sum_{u\in\mu_U^2\backslash F^\times}\sum_{a\in F_+^\times}\int_{(\mathbb{B}_f)_a} r(g,(t_1t_2^{-1},1))\phi^v(y,u)l_{r(g,(t_1t_2^{-1},1))\phi_v}(t_2,y,u)dy\\
=&\sum_{u\in\mu_U^2\backslash F^\times}\sum_{y\in B^\times/B^1}\int_{B^1(\mathbb{A}_f)} r(g,(t_1t_2^{-1},1))\phi^v(yh,u)l_{r(g,(t_1t_2^{-1},1))\phi_v}(t_2,yh,u)dh\\
=&\sum_{u\in\mu_U^2\backslash F^\times}\int_{B^1\backslash B^1(\mathbb{A}_f)}\sum_{y\in B^\times} r(g,(t_1t_2^{-1},1))\phi^v(yh,u)l_{r(g,(t_1t_2^{-1},1))\phi_v}(t_2,yh,u)dh.
\end{aligned}
$$

The integral on $B^1\backslash B^1(\mathbb{A}_f)$ essentially reduces to a finite sum. When l_{ϕ_v} is a Schwartz function, we may replace the pseudo-theta series above by the associated theta series. We will get an integral of this theta series, which gives an Eisenstein series.

Next we consider the case that $\mathbb{B}_v$ is a division algebra. In this case, B_v is a matrix algebra and the function l_{ϕ_v} above is never Schwartz in y. To resolve this problem, we take an open compact subset D of B_v such that $D_a = D\cap B(F_v)_a$ is nonempty for all a in the range

$$\{q(x): x\in\mathbb{B}_v, \phi_v(x,F_v^\times)\neq 0\}.$$

This is possible for $\phi_v\in\mathscr{S}^0(\mathbb{B}_v\times F_v^\times)$ and D depends on ϕ_v. Then we define

$$l_{\phi_v}(t_2,y,u)=\frac{1}{\mathrm{vol}(D_{q(y)})}1_D(y)\int_{(\mathbb{B}_v)_{q(y)}}\phi_v(x,u)\ l(t_2x,t_2)dx,\quad (y,u)\in B_v^\times\times F_v^\times.$$

It is easy to check that it is a Schwartz function for $(y,u)\in B_v\times F_v^\times$. And we also have

$$\int_{B(F_v)_a} l_{\phi_v}(t_2,y,u)dy=\int_{(\mathbb{B}_v)_a}\phi_v(x,u)\ l(t_2x,t_2)dx.$$

Then the argument is the same as before. We still get an integral of a non-singular pseudo-theta series. The integration of the associated theta series becomes an Eisenstein series.

Remark. The matching between $l_{\phi_v}(t_2,y,u)$ and $\phi_v(x,u)\ l(t_2x,t_2)$ so that they have the same integral looks very similar to the situation of Fundamental Lemma.

6 Pseudo-theta series and Gross–Zagier formula

In this section we prove the main result, Theorem 1.3.1. We argue that the computation and estimation of singularities we have made are enough to make the conclusion. Taking advantage of the modularity of both sides, we use some approximation to take care the terms we don't know how to compute.

Besides the linear independence of quasi-multiplicative Whittaker functions and their quasi-derivatives used in our previous paper [Zh1], the new ingredient involved in the precise proof is a theory of pseudo-theta series described in §6.1. Such series arise naturally in the computation of the derivative and the local height pairing. Roughly speaking, there are many series we don't know how to compute explicitly, and their local components are really messed. Then Lemma 6.1.1 asserts that, if a finite sum of pseudo-theta series is automorphic, then it can actually be written as a finite sum of usual theta series. The lemma simplifies the proof significantly. We expect that it also helps with the computation of derivative formula for triple product L-functions in our another joint work.

6.1 Pseudo-theta series

We use the name "pseudo-theta series" because it looks like theta series and can be compared with some theta series associated to it. Note that it is usually not automorphic.

Pseudo-theta series

Now we introduce pseudo-theta series. Let V be a positive definite quadratic space over F, and $V_0 \subset V_1 \subset V$ be two subspaces over F with induced quadratic forms. We allow V_0 to be empty. Let S be a finite set of finite places of F, and $\phi^S \in \mathscr{S}(V(\mathbb{A}^S) \times \mathbb{A}^{S\times})_{\mathrm{GO}(V_\infty)}$ be a Schwartz function with the standard infinite components.

A *pseudo-theta series* is a series of the form

$$A^{(S)}_{\phi'}(g) = \sum_{u \in \mu^2 \backslash F^\times} \sum_{x \in V_1 - V_0} \phi'_S(g,x,u) r_V(g) \phi^S(x,u), \qquad g \in \mathrm{GL}_2(\mathbb{A}).$$

We explain the notations as follows:

- The Weil representation r_V is not attached to the space V_1 but to the space V;
- $\phi'_S(g,x,u) = \prod_{v\in S} \phi'_v(g_v,x_v,u_v)$ as a product of local terms;
- For each $v \in S$, the function

  $$\phi'_v : \mathrm{GL}_2(F_v) \times (V_1 - V_0)(F_v) \times F_v^\times \to \mathbb{C}$$

  is locally constant. And it is smooth in the sense that there is an open compact subgroup K_v of $\mathrm{GL}_2(F_v)$ such that

  $$\phi'_v(g\kappa,x,u) = \phi'_v(g,x,u), \quad \forall (g,x,u) \in \mathrm{GL}_2(F_v) \times (V_1 - V_0)(F_v) \times F_v^\times,\ \kappa \in K_v.$$

- μ is a subgroup of $O_F^\times$ with finite index such that $\phi^S(x,u)$ and $\phi'_S(g,x,u)$ are invariant under the action $\alpha : (x,u) \mapsto (\alpha x, \alpha^{-2}u)$ for any $\alpha \in \mu$. This condition makes the summation well-defined.

- For any $v \in S$ and $g \in \mathrm{GL}_2(F_v)$, the support of $\phi'_S(g,\cdot,\cdot)$ in $(V_1 - V_0)(F_v) \times F_v^\times$ is bounded. This condition makes the sum convergent.

The pseudo-theta series $A^{(S)}$ sitting on the triple $V_0 \subset V_1 \subset V$ is called *non-degenerate* if $V_1 = V$, and is called *non-truncated* if V_0 is empty. It is called to be *of Whittaker type* if $\phi'_S(g,\cdot,\cdot)$ transfer according to Weil representation under the unipotent group, i.e.,

$$\phi'_S(n(b)g,x,u) = \phi'_S(g,x,u)\ \psi(buq(x)), \quad \forall b \in \mathbb{A}.$$

We use this name because, if the above is true, then $A^{(S)}$ is invariant under the left action of $N(F)$, and its first Fourier coefficient is exactly

$$\sum_{\substack{(x,u)\in\mu\backslash((V_1-V_0)\times F^\times)\\ uq(x)=1}} \phi'_S(g,x,u) r_V(g)\phi^S(x,u).$$

We call it the *Whittaker function* of $A^{(S)}$.

The pseudo-theta series $A^{(S)}$ is called *non-singular* if for each $v \in S$, the local component $\phi'_v(1,x,u)$ can be extended to a Schwartz function on $V_1(F_v) \times F_v^\times$.

Assume that $A^{(S)}_{\phi'}$ is non-singular. Then there are two usual theta series associated to $A^{(S)}$. Extending by zero we may view $\phi'_v(1,\cdot,\cdot)$ as a Schwartz function on $V_1(F_v) \times F_v^\times$ for each $v \in S$, and by restriction we may view ϕ_w as a Schwartz function on $V_1(F_w) \times F_w^\times$ for each $w \notin S$. Then the theta series

$$\theta_{A,1}(g) = \sum_{u\in\mu^2\backslash F^\times} \sum_{x\in V_1} r_{V_1}(g)\phi'_S(1,x,u) r_{V_1}(g)\phi^S(x,u)$$

is called *the first theta series associated to* $A^{(S)}_{\phi'}$. Replacing the space V_1 by V_0, we get the theta series

$$\theta_{A,0}(g) = \sum_{u\in\mu^2\backslash F^\times} \sum_{x\in V_0} r_{V_0}(g)\phi'_S(1,x,u) r_{V_0}(g)\phi^S(x,u).$$

We call it *the second theta series associated to* $A^{(S)}_{\phi'}$. We set $\theta_{A,0} = 0$ if V_0 is empty.

We introduce these theta series because the difference between $\theta_{A,1}$ and $\theta_{A,0}$ somehow approximates $A^{(S)}$.

Basic properties

Now we consider the relation between the non-singular pseudo-theta series $A^{(S)}_{\phi'}$ and its associated theta series $\theta_{A,1}$ and $\theta_{A,0}$.

We first consider the non-truncated case. Then V_0 is empty, and

$$A_{\phi'}^{(S)}(g)=\sum_{u\in\mu^2\backslash F^\times}\sum_{x\in V_1}\phi'_S(g,x,u)r_V(g)\phi^S(x,u).$$

Apparently we have $A_{\phi'}^{(S)}(1)=\theta_{A,1}(1)$. But of course we can get more.

A simple computation using Iwasawa decomposition asserts that, if ϕ_w is the standard Schwartz function on $V(F_w)\times F_w^\times$, then for any $(x,u)\in V_1(F_w)\times F_w^\times$,

$$r_V(g)\phi_w(x,u)=\begin{cases}\delta_w(g)^{\frac{d-d_1}{2}}r_{V_1}(g)\phi_w(x,u) & \text{if } w\nmid\infty;\\ \lambda_w(g)^{\frac{d-d_1}{2}}\delta_w(g)^{\frac{d-d_1}{2}}r_{V_1}(g)\phi_w(x,u) & \text{if } w\mid\infty.\end{cases}$$

Here we write $d=\dim V$ and $d_1=\dim V_1$. Here δ_w and λ_w coming from Iwasawa decomposition are explained in Section 1.6.

This result implies that,

$$A_{\phi'}^{(S)}(g)=\lambda_\infty(g)^{\frac{d-d_1}{2}}\delta(g)^{\frac{d-d_1}{2}}\theta_{A,1}(g),\qquad\forall\ g\in 1_{S'}\mathrm{GL}_2(\mathbb{A}^{S'}).$$

Here S' is a finite set consisting of finite places v such that $v\in S$ or ϕ_v is not standard.

Now we consider the general

$$A_{\phi'}^{(S)}(g)=\sum_{u\in\mu^2\backslash F^\times}\sum_{x\in V_1-V_0}\phi'_S(g,x,u)r_V(g)\phi^S(x,u).$$

We have to compare it with the difference between the theta series

$$\theta_{A,1}(g)=\sum_{u\in\mu^2\backslash F^\times}\sum_{x\in V_1}r_{V_1}(g)\phi'_S(1,x,u)r_{V_1}(g)\phi^S(x,u)$$

and the non-truncated pseudo-theta series

$$B_{\phi'}^{(S)}(g)=\sum_{u\in\mu^2\backslash F^\times}\sum_{x\in V_0}r_{V_1}(g)\phi'_S(1,x,u)r_{V_1}(g)\phi^S(x,u).$$

Note that $B^{(S)}$ is just part of $\theta_{A,1}$, where the summation is taken over V_0 but the representation is taken over V_1. By the discussion above, we should compare $B^{(S)}$ with the associated theta series

$$\theta_{B,1}(g)=\sum_{u\in\mu^2\backslash F^\times}\sum_{x\in V_0}r_{V_0}(g)\phi'_S(1,x,u)r_{V_0}(g)\phi^S(x,u).$$

But this is exactly the same as $\theta_{A,0}$. By the same argument, there exists a finite set S' of finite places such that

$$\begin{aligned}A_{\phi'}^{(S)}(g)&=\lambda_\infty(g)^{\frac{d-d_1}{2}}\delta(g)^{\frac{d-d_1}{2}}(\theta_{A,1}(g)-B_{\phi'}^{(S)}(g)),\qquad\forall\ g\in 1_{S'}\mathrm{GL}_2(\mathbb{A}^{S'});\\ B_{\phi'}^{(S)}(g)&=\lambda_\infty(g)^{\frac{d_1-d_0}{2}}\delta(g)^{\frac{d_1-d_0}{2}}\theta_{A,0}(g),\qquad\forall\ g\in 1_{S'}\mathrm{GL}_2(\mathbb{A}^{S'}).\end{aligned}$$

Our conclusion is that for any $g \in 1_{S'}\mathrm{GL}_2(\mathbb{A}^{S'})$,

$$A_{\phi'}^{(S)}(g) = \lambda_\infty(g)^{\frac{d-d_1}{2}}\delta(g)^{\frac{d-d_1}{2}}\theta_{A,1}(g) - \lambda_\infty(g)^{\frac{d-d_0}{2}}\delta(g)^{\frac{d-d_0}{2}}\theta_{A,0}(g). \tag{6.1.1}$$

By the smoothness condition of pseudo-theta series, there exists an open compact subgroup $K_{S'}$ of $\mathrm{GL}_2(F_{S'})$ such that the above identity is actually true for any $g \in K_{S'}\mathrm{GL}_2(\mathbb{A}^{S'})$.

If furthermore $A^{(S)}$ is of Whittaker type, the above remains true for any $g \in K_{S'}\mathrm{GL}_2(\mathbb{A}^{S'})$ if we replace $A^{(S)}, \theta_{A,1}, \theta_{A,0}$ by their Whittaker functions. It holds by the same argument since the Whittaker functions are just a part of their corresponding pseudo-theta series or theta series.

Now we can state our main result for pseudo-theta series.

Lemma 6.1.1. *Let $\{A_\ell^{(S_\ell)}\}_\ell$ be a finite set of non-singular pseudo-theta series of Whittaker type sitting on vector spaces $V_{\ell,0} \subset V_{\ell,1} \subset V_\ell$. Let $\{(E_i, D_i)\}$ be a finite set of pairs of Eisenstein series for some non-equivalent pairs of characters (μ_i, ν_i) and their derivations. Assume that the sum*

$$f := \sum_\ell A_\ell^{(S_\ell)}(g) + \sum_i (E_i + D_i)(g)$$

is cuspidal in $g \in \mathrm{GL}_2(\mathbb{A})$. Then

(1) $\displaystyle\sum_\ell A_\ell^{(S_\ell)} = \sum_{\ell \in L_{0,1}} \theta_{A_\ell,1}$,

(2) $\displaystyle\sum_{\ell \in L_{k,1}} \theta_{A_\ell,1} - \sum_{\ell \in L_{k,0}} \theta_{A_\ell,0} = 0, \quad \forall k \in \mathbb{Z}_{>0}$.

(3) $D_i = 0$.

Here $L_{k,1}$ is the set of ℓ such that $\dim V_\ell - \dim V_{\ell,1} = k$, and $L_{k,0}$ is the set of ℓ such that $\dim V_\ell - \dim V_{\ell,0} = k$. In particular, $L_{0,1}$ is the set of ℓ such that $V_{\ell,1} = V_\ell$.

Proof. In the equation of f, replace each $A_\ell^{(S_\ell)}$ by its corresponding combinations of theta series on the right-hand side of equation (6.1.1). After recollecting these theta series according to the powers of $\lambda_\infty(g)\delta(g)$, we end up with an equation of the following form:

$$\sum_{k=0}^{n} \lambda_\infty(g)^k \delta(g)^k f_k(g) = 0, \qquad \forall g \in K_S\mathrm{GL}_2(\mathbb{A}^S). \tag{6.1.2}$$

Here S is some finite set of finite places, K_S is an open compact subgroup of $\mathrm{GL}_2(F_S)$, and $f_0, f_1, \cdots, f_n$ are some automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ coming from combinations of f and theta series. In particular,

$$f_0 = f - \sum_{\ell \in L_{0,1}} \theta_{A_\ell,1} - \sum (E_i + D_i).$$

We will show that $f_k = 0 = D_i$ identically, which is exactly the result of (1), (2), and (3).

As all theta series f_k are defined by positive definite quadratic space, f_k has a decomposition

$$f_k = \sum_m f_{k,m}, \qquad k > 0$$

$$f_0 = \sum_m f_{0,m} + \sum D_m.$$

into a finite sum of forms and their derivations in the irreducible automorphic representations π_m of $\mathrm{GL}_2(\mathbb{A})$. The equation of the Whittaker functions corresponding to 6.1.2 is given by

$$\sum_{k,m} \lambda_\infty(g)^k \delta(g)^k W_{f_{k,m}}(g) + \sum_m W_{D_m}(g) = 0, \qquad \forall g \in K_S \mathrm{GL}_2(\mathbb{A}^S). \tag{6.1.3}$$

This is a sum of quasi-multiplicative functions on $\mathrm{GL}_2(\mathbb{A}^S)$ as $\delta(g)$ is already multiplicative. Now we can apply Lemma 4.5.1 in our previous paper [Zh1] to functions on Kirillov models

$$\alpha_{k,m}(a) := \delta^k W_{f_{k,m}} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \qquad h_m(a) := W_{D_m} \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$$

to conclude the following:

- $D_m = 0$ for all m;
- for any pair (k, m), there is another pair (k', m') such that

  $$\delta(g)^k W_{f_{k,m}}(g) + \delta(g)^{k'} W_{f_{k',m'}}(g) = 0$$

  for $g \in \mathrm{GL}_2(\mathbb{A}^S)$.

The second equation implies an equality between spherical Whittaker functions $W_m^0(g)$ for almost all unramified places v:

$$W^0_{m,v}(g) = c\delta(g)^{k'-k} W^0_{m',v}(g)$$

where c is a nonzero constant. It follows that $\pi_{m,v}$ and $\pi_{m',v}$ have the same central character. Evaluate the above identity for $g = \begin{pmatrix} \pi^m & \\ & 1 \end{pmatrix}$ to conclude that $\pi_m = \pi_{m'} \otimes |\cdot|^{(k'-k)/2}$. Thus we must have $k = k'$, $\pi_{m,v} = \pi_{m',v}$. By strong multiplicity one, we have $\pi = \pi'$ and then $f_{k,m} = 0$. $\square$

6.2 Proof of Gross–Zagier formula

Recall that

$$P(\phi, \varphi, \chi, s) = \mathrm{vol}(U_T) \left(\int_{C_U} I(s, g, r(t,1)\phi)\chi(t)dt,\ \varphi(g) \right).$$

We are going to show:

Theorem 6.2.1.

$$P'(\phi,\varphi,\chi,0) = -2\mathrm{vol}(U_T)\left(\int_{C_U}\langle Z_\phi(g)t^\circ,\ 1^\circ\rangle\chi(t)dt,\ \varphi(g)\right)$$
$$= -2\frac{\mathrm{vol}(U_T)}{|C_U|}\left(\int_{C_U}\int_{C_U}\langle Z_\phi(g)t_1^\circ,\ t_2^\circ\rangle\chi(t_1t_2^{-1})dt_1dt_2,\ \varphi(g)\right).$$

Here we write $t^\circ = t - \deg(t)\xi$.

Note that for any $c\in\mathbb{A}_f^\times$, we have

$$\begin{aligned} Z^*_\phi(cg)[t] &= \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(cg)\phi(x)_a[tx] = \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(cx)_a[tx] \\ &= \sum_{a\in F^\times}\sum_{x\in\mathbb{B}_f^\times/U} r(g)\phi(x)_a[tc^{-1}x] = Z^*_\phi(g)[c^{-1}t]. \end{aligned}$$

It already implies that $Z_\phi(cg)[t] = Z_\phi(g)[c^{-1}t]$ by modularity. It follows that

$$\int_{C_U}\langle Z_\phi(g)t^\circ,\ 1^\circ\rangle\chi(t)dt$$

has the same central character as φ, and thus the Petersson inner product in the theorem makes sense.

The theorem is equivalent to that

$$\int_{C_U}\left(I'(0,g,r(t,1)\phi) + 2\langle Z_\phi(g)t^\circ,\ 1^\circ\rangle\right)\chi(t)dt$$

is perpendicular to φ. Now we are going to compare those two kernels.

By Proposition 3.7.1, replacing ϕ by $r(t,1)\phi$, we get

$$\begin{aligned} &\mathscr{P}r(I'(0,g,r(t,1)\phi)) \\ =& -\sum_{v|\infty}\widetilde{I}'(0,g,r(t,1)\phi)(v) - \sum_{v\nmid\infty\ non-split} I'(0,g,r(t,1)\phi)(v) \\ &+ 2\sum_{(y,u)\in\mu_U\backslash E^\times\times F^\times}(\log\delta_f(g_f) + \frac{1}{2}\log|uq(y)|_f)r(g,(t,1))\phi(y,u) \\ &- \sum_v\sum_{u\in\mu_U^2\backslash F^\times, y\in E^\times} c_{\phi_v}(g,y,u)r(g,(t,1))\phi^v(y,u) - c_1\sum_{u\in\mu_U^2\backslash F^\times, y\in E^\times} r(g,(t,1))\phi(y,u) \\ &+\ \text{Eisenstein series and their derivations.} \end{aligned}$$

Here

$$\widetilde{I}'(0,g,r(t,1)\phi)(v) = 2\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\widetilde{\mathscr{K}_\phi^{(v)}}(g,(tt',t'))dt', \qquad v|\infty,$$
$$I'(0,g,r(t,1)\phi)(v) = 2\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})}\mathscr{K}_\phi^{(v)}(g,(tt',t'))dt', \qquad v\nmid\infty.$$

Now we list our computational results for the geometric part. Our place-to-place computation and comparison show that we also have

$$\begin{aligned}
\langle Z_\phi(g)t^\circ, 1^\circ\rangle &= \sum_v i_v(Z_\phi^*(g)t,1)\log N_v + \sum_v j_v(Z_\phi^*(g)t,1)\log N_v \\
&\quad + i_0(1,1)\sum_{a\in F^\times}\sum_{y\in E^\times/\mu_U} r(g,(t,1))\phi(y)_a + \langle Z_{\phi,0}(g)t,1\rangle \\
&\quad + \text{Eisenstein series and their derivations} \\
&= \sum_{v\ non-split}\log N_v \fint_{C_U}\mathscr{M}_\phi^{(v)}(g,(tt',t'))dt' + \sum_{v\notin\Sigma}\log N_v\fint_{C_U}\mathscr{N}_\phi^{(v)}(g,(tt',t'))dt' \\
&\quad + \sum_v \log N_v \fint_{C_U} j_{\bar v}(Z_\phi^*(g)tt',t')dt' \\
&\quad + i_0(1,1)\sum_{a\in F^\times}\sum_{y\in E^\times/\mu_U} r(g,(t,1))\phi(y)_a + \langle Z_{\phi,0}(g)t,1\rangle \\
&\quad + \text{Eisenstein series and their derivations.}
\end{aligned}$$

Since the archimedean parts are standard, we see that the average integrals

$$\fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})} = \fint_{Z(\mathbb{A})T(F)\backslash T(\mathbb{A})/U_T} = \fint_{T(F)\backslash T(\mathbb{A})/U_T} = \fint_{C_U}.$$

Furthermore, C_U is finite and the integral reduces to a finite sum.

In summary, we have proved that

$$\mathscr{P}r'\int_{C_U}\left(I'(0,g,r(t,1)\phi) + 2\langle Z_\phi(g)t^\circ,\ 1^\circ\rangle\right)\chi(t)dt = \int_{C_U}\mathscr{D}_\phi(g,(t,1))\chi(t)dt,$$

where

$$\mathscr{D}_\phi(g,(t,1)) = \mathscr{P}r(I'(0,g,r(t,1)\phi)) + 2\langle Z_\phi(g)t^\circ,\ 1^\circ\rangle$$

is a sum of non-singular pseudo-theta series, Eisenstein series and their derivations.

Here is a checklist of the comparison:

(1) For $v|\infty$, Proposition 5.1.1 shows that

$$\widetilde{\mathscr{K}}_\phi^{(v)}(g,(t_1,t_2)) - \mathscr{M}_\phi^{(v)}(g,(t_1,t_2)) = 0.$$

(2) For v finite and non-split in E, we have showed that

$$\mathscr{K}_\phi^{(v)}(g,(t_1,t_2)) - \mathscr{M}_\phi^{(v)}(g,(t_1,t_2))\log N_v$$

is a nonsingular pseudo-theta series and vanishes for almost all v. This pseudo-theta series is non-degenerate. See Proposition 5.2.6 and Proposition 5.2.7(1) for the supersingular case and Proposition 5.3.2 for the superspecial case.

(3) For $v \notin \Sigma$, we have showed that the difference

$$\mathscr{N}_\phi^{(v)}(g,(t_1,t_2)) + \sum_{(y,u)\in\mu_U\backslash E^\times\times F^\times} (\log\delta_f(g_f) + \frac{1}{2}\log|uq(y)|_f) r(g,(t_1,t_2))\phi(y,u)$$

is a nonsingular pseudo-theta series and vanishes for almost all v. This pseudo-theta series is degenerate. See Proposition 5.2.6 and Proposition 5.2.7(2) for the supersingular case and Proposition 5.4.5 and Proposition 5.4.6 for the ordinary case.

(4) For all finite v, we have showed that $j_{\bar{v}}(Z_\phi^*(g)tt',t')$ is an integral of a nonsingular pseudo-theta series on $B(v)$ if ϕ_v. This pseudo-theta series is non-degenerate. The integration of the associated theta series gives a Siegel–Einseisten series. See Section 5.5.

(5) The rest of terms are explicitly expressed as non-singular pseudo-theta series. Some are non-singular automatically, and some are non-singular if $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$.

(6) The Einsenstein series and their derivation comes from Proposition 3.6.3 for holomorphic projection and Proposition 4.6.1 for the height pairing related to the Hodge class ξ. See §4.4.4 in our previous paper [Zh1] for a general explanation of derivations.

Only finitely many terms are left after the comparison. Note that the integral on C_U is just a finite sum. Then we are in the situation to apply Lemma 6.1.1 to

$$\int_{C_U} \mathscr{D}_\phi(g,(t,1))\chi(t)dt = \sum A_\ell + \sum_i (E_i + D_i).$$

Since their sum is modular and cuspidal, it is equal to a finite sum of usual theta series and Eisenstein series by Lemma 6.1.1. These theta series are coming from $\mathscr{K}_\phi^{(v)}(g,(t_1,t_2)) - \mathscr{M}_\phi^{(v)}(g,(t_1,t_2))\log N_v$ and $j_{\bar{v}}(Z_\phi^*(g)tt',t')$, the only non-degenerate pseudo-theta series in the list. Note that the quadratic spaces for both of them are the nearby quaternion algebra $B(v)$. Therefore, we end up with

$$\mathscr{D}_\phi(g,(t,1)) = \fint_{C_U} \sum_v \theta_{\phi^{(v)}}(g,(tt',t'))dt' + \text{Eisenstein Series}.$$

We understand that the equality is true after integrating against $\chi(t)$ on C_U, and that the theta series is given by

$$\theta_{\phi^{(v)}}(g,(t_1,t_2)) = \sum_{u\in\mu_U^2\backslash F^\times} \sum_{y\in B(v)} r(g,(t_1,t_2))\phi'_v(y_v,u_v) r(g,(t_1,t_2))\phi^v(y,u)$$

for some Schwartz function $\phi^{(v)} = \phi_v^{(v)} \otimes \phi^v \in \mathscr{S}(B(v)_{\mathbb{A}} \times \mathbb{A}^\times)$ with $\phi_v^{(v)} \in \mathscr{S}(B(v)_v \times F_v^\times)$.

Therefore, we get that

$$\left(\int_{C_U} \chi(t)\mathscr{D}_\phi(g,(t,1))dt, \quad \varphi(g)\right)$$
$$=\sum_v \left(\int_{C_U}⨍_{C_U} \chi(t)\theta_{\phi^{(v)}}(g,(tt',t'))dt'dt, \quad \varphi(g)\right).$$

Now the vanishing of the right-hand side following from the criterion of the existence of local χ-linear vectors by the result of Tunnell [Tu] and Saito [Sa], Proposition 1.1.1.

Note that to control the singularities for some ramified non-archimedean place v, we have assumed that $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$. But we can easily extend the result to all Schwartz function $\phi_v \in \mathscr{S}(\mathbb{B}_v \times F_v^\times)$. We have showed that the main theorem is true for all pairs (φ, ϕ) with $\varphi \in \pi$ and $\phi = \otimes_v \phi_v$ such that ϕ_v is standard for good v, and $\phi_v \in \mathscr{S}^0(\mathbb{B}_v \times F_v^\times)$ for bad v. By Proposition 3.3.1, all theta liftings θ_ϕ^φ given by such pairs span the space $\pi' \otimes \bar{\pi}'$. On the other hand, the result of the main theorem depends only on the theta lifting θ_ϕ^φ. We conclude that the main theorem is true for all φ, ϕ.

References

[Gr] B. Gross: *Heegner points and representation theory.* Heegner points and Rankin L-series, 37–65, Math. Sci. Res. Inst. Publ., 49, Cambridge Univ. Press, Cambridge, 2004.

[GZ] B. Gross; D. Zagier: *Heegner points and derivatives of L-series.* Invent. Math. 84 (1986), no. 2, 225–320.

[KM1] S. Kudla; J. Millson: *The theta correspondence and harmonic forms. I.* Math. Ann. 274 (1986), no. 3, 353–378.

[KM2] S. Kudla; J. Millson: *The theta correspondence and harmonic forms. II.* Math. Ann. 277 (1987), no. 2, 267–314.

[KM3] S. Kudla; J. Millson: *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables.* Inst. Hautes Études Sci. Publ. Math. No. 71 (1990), 121–172.

[KRY] S. Kudla; M. Rapoport; T. Yang: *Modular forms and special cycles on Shimura curves.* Annals of Mathematics Studies, 161. Princeton University Press, Princeton, NJ, 2006.

[Ku1] S. Kudla: *Algebraic cycles on Shimura varieties of orthogonal type.* Duke Math. J. 86 (1997), no. 1, 39-78.

[Ku2] S. Kudla: *Central derivatives of Eisenstein series and height pairings.* Ann. of Math. (2) 146 (1997), no. 3, 545–646.

[Sa] H. Saito: *On Tunnell's formula for characters of* GL(2). Compositio Math. 85 (1993), no. 1, 99–108.

[Tu] J. Tunnell: *Local ϵ-factors and characters of* GL(2). Amer. J. Math. 105 (1983), no. 6, 1277–1307.

[Wa] J. Waldspurger: *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie.* Compositio Math. 54 (1985), no. 2, 173–242.

[We] A. Weil: *Sur la formule de Siegel dans la théorie des groupes classiques.* Acta Math. 113 (1965) 1–87.

[YZZ] X. Yuan, S. Zhang, W. Zhang: *The Gross-Kohnen-Zagier theorem over totally real fields.* Preprint.

[Zh1] S. Zhang: *Heights of Heegner points on Shimura curves.* Ann. of Math. (2) 153 (2001), no. 1, 27–147.

[Zh2] S. Zhang: *Gross-Zagier formula for* GL_2. Asian J. Math. 5 (2001), no. 2, 183–290.

[Zh3] S. Zhang: *Gross-Zagier formula for* GL(2). II. Heegner points and Rankin L-series, 191–214, Math. Sci. Res. Inst. Publ., 49, Cambridge Univ. Press, Cambridge, 2004.