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Weil Representation, Howe Duality, and the Theta Correspondence

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These are some expository lectures given in Montreal on the Weil representation. The aim of these lectures was to introduce the audience to this important representation and to show how it has been useful in constructing representations of p-adic groups and automorphic forms via dual reductive pairs. These lectures contain no serious proofs, for given the limitations of time, the inclusion of all the technical details would only have served to detract from the global structure. We refer the reader to the nice book of Moeglin, Vigneras, Waldspurger [**MVW**] for all the proofs in the local non-Archimedean case, and the basic papers of Howe [**H2**] and [**H3**] for the real theory.

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1. Heisenberg group

In this section k will denote a non-Archimedean local field which will never be of characteristic 2. All the representations will be over complex numbers and smooth in the sense that every vector in the representation space is fixed by a compact open subgroup. For any finite dimensional vector space V over k, $\mathcal{S}(V)$ will denote the space of locally constant compactly supported functions on V.

Let W be a finite dimensional vector space over k with a non-degenerate alternating form $\langle\ ,\ \rangle$. Such a vector space will be called a symplectic space. Its dimension is even and we denote it by $2n$. The Heisenberg group $H(W)$ associated to the symplectic space W is a non-trivial central extension of W by k and is defined to be the group of pairs

$$\{(w,t)|w \in W, t \in k\},$$

with the law of multiplication

$$(w_1,t_1)(w_2,t_2) = (w_1+w_2, t_1+t_2+\frac{1}{2}\langle w_1,w_2\rangle).$$

The Heisenberg group $H(W)$ clearly sits in the exact sequence

$$0 \to k \to H(W) \to W \to 0.$$

The commutator subgroup of $H(W)$ is k and all the one dimensional representations of $H(W)$ factor through W. We construct below an infinite dimensional smooth

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representation of $H(W)$. For this fix a decomposition of W as $W = W_1 \oplus W_2$ where W_1 and W_2 are maximal totally isotropic subspaces of W (i.e., subspaces on which the alternating form is identically zero). Such a decomposition will be called a complete polarisation of W. The representation depends on the choice of an additive character ψ of k which will be fixed in all these lectures.

Define the representation ρ_ψ of the Heisenberg group $H(W)$ on $\mathcal{S}(W_1)$ as follows:

$$\begin{aligned} \rho_\psi(w_1)f(x) &= f(x+w_1) && \text{for all} \quad x, w_1 \in W_1, \\ \rho_\psi(w_2)f(x) &= \psi(\langle x, w_2\rangle)f(x) && \text{for all} \quad x \in W_1, w_2 \in W_2, \\ \rho_\psi(t)f(x) &= \psi(t)f(x) && \text{for all} \quad t \in k, x \in W_1. \end{aligned}$$

It can be easily checked that this gives a smooth representation of $H(W)$. This representation of $H(W)$ is called the Schrödinger representation.

REMARK 1.1. If $\bar{\psi}$ denotes the character $\bar{\psi}(x) = \psi(-x)$, then $\rho_{\bar{\psi}}$ is the smooth dual of ρ_ψ. To see this observe that

$$(f_1, f_2) \to \int_{W_1} f_1 f_2 dw, \quad f_1, f_2 \in \mathcal{S}(W_1)$$

is a non-degenerate bilinear form on $\rho_\psi \otimes \rho_{\bar{\psi}}$.

There is another model of this representation, called the lattice model which is also quite useful. In fact we define a representation which is more general than both of these.

Let $A \subset W$ be a closed subgroup and define $A_k = A \times k \subset H(W)$. It is a subgroup of $H(W)$. Assume now that $A = A^\perp$ where

$$A^\perp = \{y \in W \mid \psi(\langle x, y\rangle) = 1, \quad \forall x \in A\}.$$

In this case the character ψ of k can be extended to A_k. Fix such an extension and call it ψ_A.

Let S_A be the space of functions on $H(W)$ such that

(i) $f(ah) = \psi_A(a)f(h)$ for all $a \in A_k, h \in H(W)$.
(ii) $f(hl) = f(h)$ for all $l \in L$, a lattice in W, and $h \in H(W)$.

By a simple calculation one can show that functions in S_A are compactly supported modulo A_k. Clearly the Heisenberg group operates on S_A via right translations and this representation is smooth because of (ii).

Since k operates via the character ψ on functions in S_A, one can realise this representation also on a certain space of functions on $H(W)/k$. To do this, let f be a function on $H(W)$ such that $f(ht) = \psi(t)f(h)$ for $t \in k$, and define a function $\tilde{f}$ on W by $\tilde{f}(w) = f(w,t)\psi(t)^{-1}$ for any $t \in k, w \in W$. Conversely given $\tilde{f}$, the same equation can be used to define the function f on $H(W)$. Under this isomorphism, the space S_A gets identified to the space of functions $\tilde{f}$ in $\mathcal{S}(W)$ such that $\tilde{f}(a+w) = \psi(\frac{1}{2}\langle w, a\rangle)\tilde{f}(w)$ for all $a \in A$, and $b \in W \subset H(W)$ acts on this space of functions $\tilde{f}$ by, $(b \cdot \tilde{f})(w) = \psi(\frac{1}{2}\langle w, b\rangle)\tilde{f}(b+w)$.

PROPOSITION 1.2. *Each S_A is an irreducible representation of $H(W)$.*

We now give some examples of closed subgroups A with $A^\perp = A$

(1) $A = W_1$, a maximal totally isotropic subspace of W.
(2) $A = L = \sum_{i=1}^n \mathcal{O}_k e_i \oplus \sum_{i=1}^n \mathcal{O}_k f_i$ where $\{e_i, f_i\}$ is a symplectic basis of W i.e. $\langle e_i, e_j\rangle = 0, \langle f_i, f_j\rangle = 0, \langle e_i, f_j\rangle = \delta_{ij}$, and $\mathcal{O}_k$ is the ring of integers of

k with uniformising parameter π_k. Then if the conductor of ψ is $\mathcal{O}_k$ i.e., $\psi(\mathcal{O}_k) = 1$ but $\psi(\pi_k^{-1}\mathcal{O}_k) \not\equiv 1$, $L^{\perp} = L$.

It is easy to see that in case (1), we get the Schrödinger model of the representation of the Heisenberg group discussed earlier, and the second case gives what are called lattice models.

We have now constructed several smooth irreducible representations of the Heisenberg group on which the centre acts by the character ψ. All these representations are isomorphic. In fact one has the following basic theorem.

THEOREM 1.3 (STONE, VON NEUMANN). *The Heisenberg group $H(W)$ has a unique irreducible smooth representation on which k operates via the character ψ.*

We will denote this unique irreducible smooth representation of $H(W)$ by $(\rho_\psi, \mathcal{S})$.

2. Metaplectic group and the Weil representation

The main theme of these lectures is not the representation $(\rho_\psi, \mathcal{S})$ of the Heisenberg group constructed in the last section but rather a (projective)-representation of the symplectic group which is constructed using intertwining operators of this representation of the Heisenberg group. To define this, observe that the symplectic group $Sp(W)$ of W (i.e. the automorphisms of W preserving the alternating form $\langle\ ,\ \rangle$) operates on $H(W)$ by $g \cdot (w,t) = (gw,t)$. Clearly this action is trivial on the centre of $H(W)$ and therefore by the uniqueness theorem of Stone and von Neumann, there is an operator $\omega_\psi(g)$ (unique up to scaling) on $\mathcal{S}$ such that

$$(*) \qquad \rho_\psi(gw,t) \cdot \omega_\psi(g) = \omega_\psi(g) \cdot \rho_\psi(w,t), \quad \text{for all} \quad (w,t) \in H(W).$$

Now define

$$\widetilde{Sp}_\psi(W) = \{(g, \omega_\psi(g)) \quad \text{such that} \quad (*) \quad \text{holds}\}.$$

Then $\widetilde{Sp}_\psi(W)$ is a group under pointwise multiplication, called the metaplectic group, and fits in the following exact sequence:

$$0 \to \mathbb{C}^* \to \widetilde{Sp}_\psi(W) \xrightarrow{p} Sp(W) \to 0.$$

The metaplectic group comes equipped with a natural representation obtained by projection on the second factor $(g, \omega_\psi(g)) \to \omega_\psi(g) \in \mathrm{Aut}(\mathcal{S})$. This representation of the metaplectic group is called the Weil representation or the metaplectic representation or the oscillator representation.

THEOREM 2.1. *The map p restricted to the commutator subgroup $\widehat{Sp}_\psi(W) = [\widetilde{Sp}_\psi(W), \widetilde{Sp}_\psi(W)]$ of $\widetilde{Sp}_\psi(W)$ is a surjection onto $Sp(W)$ with a kernel of order 2. In particular,*

$$\widetilde{Sp}_\psi(W) = \widehat{Sp}_\psi(W) \times_{\mathbb{Z}/2} \mathbb{C}^*.$$

Moreover, the two sheeted covering $\widehat{Sp}_\psi(W)$ of $Sp(W)$ is independent of the additive character ψ but the Weil representation restricted to $\widehat{Sp}_\psi(W)$ does depend on ψ.

We now give an explicit model, called the Schrödinger model, of the metaplectic representation over a local field. For this, let $W = W_1 \oplus W_2$ be a complete polarisation of W. We will write elements of $Sp(W)$ as matrices with respect to the basis $\{e_1, e_2, \cdots, e_n, f_1, \cdots, f_n\}$ where $e_i \in W_1$ and $f_i \in W_2$ and $\langle e_i, f_j \rangle = \delta_{i,j}$.

One can easily check that

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \quad \text{belongs to} \quad Sp(W) \text{ if } A \in GL(W_1),$$

and the action of this matrix in the metaplectic representation on $\mathcal{S}(W_1)$ is given by

$$\omega_\psi \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f(x) = |\det A|^{1/2} f({}^tAx),$$

i.e., the operator so defined has the property $(*)$ above. (Observe that since $g \to \omega_\psi(g)$ is a homomorphism on such matrices, the metaplectic covering splits on the subgroup $\left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} | A \in GL(W_1) \right\}$ of $Sp(W)$). The purpose of the scaling factor $|\det A|^{1/2}$ in the above action is to make the action of $GL(W_1)$ unitary for the hermitian structure on $\mathcal{S}(W_1)$: $(f_1, f_2) = \int_{W_1} f_1 \bar{f}_2 dw$.

We have

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \in Sp(W) \quad \text{if and only if} \quad B = {}^tB,$$

and the action of this matrix is given by

$$\omega_\psi \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(x) = \psi\left(\frac{{}^txBx}{2}\right) f(x).$$

Finally, we have

$$\omega_\psi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f(x) = \gamma \hat{f}(x),$$

where γ is an 8-th root of unity to be described below and $\hat{f}$ is the Fourier transform,

$$\hat{f}(x) = \int_{k^n} f(y) \psi\left(\sum_{i=1}^n x_i y_i\right) dy,$$

the Haar measure dy being chosen so that $\hat{\hat{f}}(x) = f(-x)$.

We define the γ-factor $\gamma(Q, \psi)$ more generally for any quadratic form Q on k^n to be $\gamma(Q, \psi) = g(Q, \psi)/|g(Q, \psi)|$ where $g(Q, \psi)$ is defined as follows. (Our γ will be $\gamma(Q, \psi)$ for $Q = \sum_{i=1}^n x_i^2$.)

$$g(Q, \psi) = P.V. \int_{k^n} \psi(Q(x)) dx \stackrel{\text{def}}{=} \lim_{m \to \infty} \int_{\pi^{-m} \mathcal{O}_k^n} \psi(Q(x)) dx,$$

where dx is now chosen so that Fourier inversion holds for $B(x, y) = \frac{1}{2}\big(Q(x+y) - Q(x) - Q(y)\big)$.

REMARK 2.2. One might wonder why this care about the 8-th root of unity factor γ when the intertwining operator $\omega_\psi \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right)$ is defined only up to scalar anyway. This has to do with the fact that the metaplectic group $\widetilde{Sp}_\psi(W)$ contains $\widehat{Sp}(W)$ which is a two sheeted covering of $Sp(W)$, and therefore the scaling factor can be normalised up to an ambiguity of sign.

REMARK 2.3. It can be proved that for the function $\psi_Q(x) = \psi(Q(x))$ on k^n, $\widehat{\psi_Q} = \gamma \psi_Q^{-1}$ in the sense of distributions.

REMARK 2.4. The construction of the Heisenberg group and the uniqueness theorem of Stone and von Neumann is also true in the case of finite fields. Therefore

as above, one can construct a projective representation of dimension q^n of $Sp(n, \mathbb{F}_q)$. This projective representation in fact lifts to an ordinary representation of $Sp(n, \mathbb{F}_q)$ (in a unique way except if $n = 1, q = 3$). The representation of $Sp(n, \mathbb{F}_q)$ so obtained depends on the choice of the additive character ψ of $\mathbb{F}_q$, and the representation associated to ψ and $\psi_a(x) = \psi(ax)$ are isomorphic if and only if a is a square in $\mathbb{F}_q$.

REMARK 2.5. Representation of the Heisenberg group and of the metaplectic group can be combined to give a representation of the semi-direct product $H(W) \rtimes \widetilde{Sp}_\psi(W)$ where the semi-direct product is via the natural action of $Sp(W)$ on $H(W)$.

2.6 Lattice Model of the Weil Representation. Let A be a finitely generated $\mathcal{O}_k$-module of maximal rank in W such that $A^\perp = A$. Then in the lattice model of the representation of the Heisenberg group on functions on W, $(g, M[g]) \in \widetilde{Sp}_\psi(W)$ for the operator $M[g]$:

$$(M[g]f)(w) = \sum_{a \in \frac{A}{gA \cap A}} \psi\left(\frac{\langle a, w\rangle}{2}\right) f(g^{-1}(a + w)).$$

In particular for K, the stabiliser of A, the above formula simplifies to

$$(M[g]f)(w) = f(g^{-1}w).$$

Observe now that this K is a maximal compact subgroup of $Sp(W)$, and since the above is clearly a representation of K (instead of a projective representation), the metaplectic covering splits on K.

Moreover, from the above action of K it is clear that the characteristic function of A is invariant under K and it is easy to see that it is the unique vector invariant under K.

REMARK 2.7. For $W = W^1 \oplus W^2$, the direct sum of two symplectic spaces, let $W^1 = W_1^1 \oplus W_2^1$, $W^2 = W_1^2 \oplus W_2^2$ and $W = (W_1^1 \oplus W_1^2) \oplus (W_2^1 \oplus W_2^2)$ be complete polarisations. One has representations $(\rho_\psi, \mathcal{S}(W_1^1))$ of $H(W^1)$, $(\rho_\psi, \mathcal{S}(W_1^2))$ of $H(W^2)$, and $(\rho_\psi, \mathcal{S}(W_1^1 \oplus W_1^2))$ of $H(W)$. Since $\mathcal{S}(W_1^1 \oplus W_1^2) = \mathcal{S}(W_1^1) \otimes \mathcal{S}(W_1^2)$, it follows that the representation $(\rho_\psi, \mathcal{S}(W_1^1 \oplus W_1^2))$ of $H(W)$ under the mapping $H(W^1) \times H(W^2) \to H(W)$ where $(w_1, t_1) \times (w_2, t_2) \in H(W^1) \times H(W^2)$ goes to $(w_1 + w_2, t_1 + t_2)$ becomes the tensor product of representations $(\rho_\psi, \mathcal{S}(W_1^1))$ of $H(W^1)$ and $(\rho_\psi, \mathcal{S}(W_1^2))$ of $H(W^2)$. Clearly the intertwining operator corresponding to $g \in Sp(W^1)$ acting on the first variable is an intertwining operator for g thought of as an element of $Sp(W^1 \oplus W^2)$. This implies that the metaplectic covering of $Sp(W^1 \oplus W^2)$ restricted to $Sp(W^1)$ is the metaplectic covering of $Sp(W^1)$, and that the metaplectic representation of $Sp(W^1 \oplus W^2)$ restricted to $Sp(W^1) \times Sp(W^2)$ is the tensor product of their metaplectic representations.

REMARK 2.8. Suppose that W is a symplectic space and V is a non-degenerate quadratic space. Then the product of the bilinear forms gives a symplectic structure on $W \otimes V$. Suppose that the quadratic form on V is $\sum_{i=1}^m \alpha_i x_i^2$; then as a symplectic space

$$W \otimes V = \sum_{i=1}^m \alpha_i W.$$

Therefore, from Remark 2.7, the restriction of the metaplectic representation of $Sp(W \otimes V)$ to $\prod_{i=1}^m Sp(\alpha_i W)$ is the tensor product of the metaplectic representations of $Sp(\alpha_i W)$. Since the metaplectic representation of $Sp(\alpha W)$ associated to

the character ψ is isomorphic to the metaplectic representation of $Sp(W)$ associated to the character $\psi_\alpha(x) = \psi(\alpha x)$, the restriction of the metaplectic representation of $Sp(W \otimes V)$ to $Sp(W)$ is isomorphic to the tensor product of the metaplectic representations of $Sp(W)$ associated to the characters ψ_{α_i}.

REMARK 2.9. For V and W as in the previous remark, $O(V) \subset Sp(W \otimes V)$ in the obvious way. Let $W = W_1 \oplus W_2$ be a complete polarisation of W. Then $W \otimes V = W_1 \otimes V \oplus W_2 \otimes V$ is a complete polarisation for $W \otimes V$. Recall now that the Schrödinger model of the metaplectic representation of $\widetilde{Sp}(W \otimes V)$ is obtained on the compactly supported locally constant functions on $W_1 \otimes V$. Now $O(V)$ operates on $\mathcal{S}(W_1 \otimes V)$ in a natural way, and it is easy to see from the explicit formulae for the representation of the Heisenberg group that this action is an intertwining operator; therefore we conclude that

(i) The metaplectic covering of $Sp(W \otimes V)$ splits on the subgroup $O(V)$.
(ii) The metaplectic representation of $Sp(W \otimes V)$ restricted to $O(V)$ is the natural representation of $O(V)$ on functions on $W_1 \otimes V$ (at least, up to a character of $O(V)$).

REMARK 2.10. Similarly if $V = V_1 \oplus V_2$ where V_1 and V_2 are maximal totally isotropic subspaces of the quadratic space V, $W \otimes V = W \otimes V_1 \oplus W \otimes V_2$ is a complete polarisation of $W \otimes V$. We conclude as in the previous remark that the metaplectic covering of $Sp(W \otimes V)$ splits on $Sp(W)$, and the metaplectic representation of $Sp(W \otimes V)$ restricted to $Sp(W)$ is the natural representation of $Sp(W)$ on $\mathcal{S}(W \otimes V_1)$.

REMARK 2.11. By Remark 1.1, the dual of the metaplectic representation ω_ψ of $\widetilde{Sp}(W)$ associated to the character $\psi(x)$ is the metaplectic representation of $\widetilde{Sp}(W)$ associated to the character $\psi(-x)$. Therefore by Remarks 2.8 and 2.10,

$$\omega_\psi \otimes \omega_\psi^* \cong \mathcal{S}(W),$$

where $Sp(W)$ operates on $S(W)$ in the natural way (observe that $\omega_\psi \otimes \omega_\psi^*$ is a representation of $Sp(W)$). This relation is specially useful for the Weil representation of $Sp(n, \mathbb{F}_q)$ where the Weil representation is self-dual if -1 is a square in $\mathbb{F}_q$ (see Remark 2.4), i.e. for $q \equiv 1 \bmod 4$, and the above relation can be used to calculate the character of the Weil representation up to sign.

3. Dual reductive pairs

A dual reductive pair is a pair of subgroups (G, G') in a symplectic group $Sp(W)$ such that

(i) G is the centraliser of G' in $Sp(W)$, and G' is the centraliser of G in $Sp(W)$.
(ii) The actions of G and G' are completely reducible on W (recall that an action is called completely reducible if every invariant subspace has an invariant complement).

A dual reductive pair (G, G') in $Sp(W)$ will be called irreducible if one can't decompose W as the direct sum of two symplectic subspaces each of which is invariant under both G and G'.

REMARK 3.1. If $(G_1, G_1') \subset Sp(W^1)$ and $(G_2, G_2') \subset Sp(W^2)$ are dual reductive pairs then $(G_1 \times G_2, G_1' \times G_2')$ is a dual reductive pair in $Sp(W^1 \oplus W^2)$.

REMARK 3.2. Every dual reductive pair is constructed from irreducible ones by repeating the process in Remark 3.1.

We will give a general method of constructing irreducible dual reductive pairs. For this we begin by defining a hermitian form on a vector space over a division algebra with involution.

Let D be a division algebra over k and τ an involution of D over k (i.e. $\tau(x+y)=\tau(x)+\tau(y), \tau(xy)=\tau(y)\tau(x), \tau^2(x)=x$ for all $x,y\in D$, and $\tau(a)=a$ for all $a\in k$). Let V be a right vector space over D with a k-bilinear form $H: V\times V\to D$ such that

(i) $H(v_1d_1, v_2d_2)=\tau(d_1)H(v_1,v_2)d_2$, for all $v_1,v_2\in V$ and $d_1,d_2\in D$.
(ii) $\tau(H(v_1,v_2))=\epsilon H(v_2,v_1)$ for all $v_1,v_2\in V$, where $\epsilon=\pm 1$.

A right D-vector space V together with a bilinear form H having the properties (i) and (ii) above is called an ϵ-hermitian space.

The group of D-automorphisms of V preserving the ϵ-hermitian form H is called the unitary group associated to the ϵ-hermitian space V, and is denoted by $U(V,H)$ (or $U(V)$ for short).

REMARK 3.3. If k is a local field then any division algebra over k with an involution is at most 4-dimensional over its centre. Moreover, if the involution is non-trivial on the centre then the division algebra is commutative.

REMARK 3.4. If $D=k$, then for $\epsilon=1$, $U(V,H)$ is the usual orthogonal group and for $\epsilon=-1$, $U(V,H)$ is the symplectic group. If $D=K=k(\sqrt{d})$ is a quadratic field extension of k, and τ is the non-trivial Galois automorphism of K/k, $U(V,H)$ is what is generally called a unitary group. In this case one can multiply an ϵ-hemitian form by $\sqrt{d}$ to go from hermitian to skew-hermitian and vice-versa without changing the unitary group. More generally, multiplying an ϵ-hermitian form by $a\in D$ such that $\tau(a)=-a$, takes the ϵ-hermitian form to a $-\epsilon$-hermitian form (for $\tau'=a\tau a^{-1}$).

REMARK 3.5. Let D be a quaternion division algebra over k with the standard involution. Then if $\epsilon=1$, $U(V,H)$ is a form of the symplectic group and if $\epsilon=-1$, the $U(V,H)$ is a form of the orthogonal group.

REMARK 3.6. The unitary groups $U(V,H)$ for ϵ-hermitian forms H constructed above, together with general linear groups over division algebras, constitute what are called the classical groups.

Classification theorem for irreducible dual reductive pairs. The irreducible dual reductive pairs (G,G') in $Sp(W)$ are of two types:

Type 1. The action of $G\cdot G'$ on W is irreducible.

In this case there exists a division algebra D/k with an involution τ, and for $i=1,2$ spaces W_i over D and ϵ_i-hermitian forms H_i on W_i with $\epsilon_1\epsilon_2=-1$ such that $W\cong W_1\otimes_D W_2$, and the alternating bilinear form on $W\cong W_1\otimes_D W_2$ is given by $\langle w_1\otimes w_2, w_1'\otimes w_2'\rangle=\mathrm{tr}_{D/k}(H_1(w_1,w_1')H_2(w_2,w_2'))$, and the pair (G,G') is $(U(W_1),U(W_2))$.

Type 2. The action of $G\cdot G'$ on W is reducible.

In this case there exists a division algebra D/k and a right D-vector space W_1 and a left D-vector space W_2 such that $W=W_1\otimes_D W_2\oplus(W_1\otimes_D W_2)^*$ with the natural symplectic structure, and $(G,G')=(\mathrm{Aut}_D(W_1),\mathrm{Aut}_D(W_2))$ with the natural action of $\mathrm{Aut}_D(W_1)$ and $\mathrm{Aut}_D(W_2)$ on $W_1\otimes_D W_2$ and its dual, and therefore on W.

4. Howe duality

We begin with the following general lemma, [**MVW**, Lemma II.5].

LEMMA 4.1. *If two elements in $Sp(W)$ commute then their arbitrary lifts in $\widetilde{Sp}_\psi(W)$ also do.*

Now for a closed subgroup $H \subset Sp(W)$, let $\widetilde{H} \subset \widetilde{Sp}_\psi(W)$ be the full inverse image of H in the metaplectic group. Let $R_\psi(\widetilde{H})$ be the set of smooth irreducible representations π of $\widetilde{H}$ such that

$$\mathrm{Hom}_{\widetilde{H}}(\mathcal{S}, \pi) \neq 0,$$

where $(\omega_\psi, \mathcal{S})$ is the metaplectic representation of $\widetilde{Sp}_\psi(W)$.

Note that for a dual reductive pair (G, G') in $Sp(W)$, the subgroups $\widetilde{G}$ and $\widetilde{G'}$ of $\widetilde{Sp}_\psi(W)$ commute by Lemma 4.1, and there is a surjective map from $\widetilde{G} \times \widetilde{G'}$ to $\widetilde{G \cdot G'}$. Therefore a representation of $\widetilde{G \cdot G'}$ is a tensor product $\pi \otimes \pi'$ of representations π of $\widetilde{G}$ and π' of $\widetilde{G'}$. Clearly if $\pi \otimes \pi'$ belongs to $R_\psi(\widetilde{G \cdot G'})$ then π belongs to $R_\psi(\widetilde{G})$ and π' to $R_\psi(\widetilde{G'})$. It follows that there is a natural map from $R_\psi(\widetilde{G \cdot G'})$ to $R_\psi(\widetilde{G}) \times R_\psi(\widetilde{G'})$.

CONJECTURE 1 (HOWE). For a dual reductive pair (G, G'), the image of the mapping from $R_\psi(\widetilde{G \cdot G'})$ to $R_\psi(\widetilde{G}) \times R_\psi(\widetilde{G'})$ constructed above is the graph of a bijective correspondence between $R_\psi(\widetilde{G})$ and $R_\psi(\widetilde{G'})$.

Actually there is a slightly more refined conjecture which takes into account the multiplicities too. To state it, define for an irreducible representation π of $\widetilde{G}$

$$\mathcal{S}(\pi) = \bigcap \ker \alpha,$$

where the intersection is taken over all the possible homomorphisms α from the Weil representation $(\omega_\psi, \mathcal{S})$ onto π, and let

$$\mathcal{S}[\pi] = \mathcal{S}/\mathcal{S}(\pi).$$

Note that $\mathcal{S}[\pi]$ is the largest quotient of $\mathcal{S}$ which is π-isotypical. Since $\widetilde{G'}$ commutes with $\widetilde{G}$, $\mathcal{S}[\pi]$ is also a $\widetilde{G'}$ module and therefore a $\widetilde{G} \times \widetilde{G'}$-module. We now have the following general lemma.

LEMMA 4.2. *Let τ be a smooth representation of the product $G_1 \times G_2$ of two locally compact totally disconnected groups, and let τ_1 be a smooth irreducible admissible representation of G_1. If the intersection of the kernels of all the G_1-homomorphisms of τ onto τ_1 is trivial, then $\tau \cong \tau_1 \otimes \tau_2$ for a smooth representation τ_2 of G_2.*

From the above lemma, there is a smooth representation $\sigma_0(\pi)$ of $\widetilde{G'}$ such that $\mathcal{S}[\pi] \cong \pi \otimes \sigma_0(\pi)$ as a $\widetilde{G} \times \widetilde{G'}$-module.

CONJECTURE 2 (HOWE). The representation $\sigma_0(\pi)$ of $\widetilde{G'}$ has a unique irreducible quotient denoted by $\sigma(\pi)$, and the mapping $\pi \to \sigma(\pi)$ is a bijection between $R_\psi(\widetilde{G})$ and $R_\psi(\widetilde{G'})$.

REMARK 4.3. The Howe conjecture is now proved for non-Archimedean local fields of residue characteristic not 2. Howe himself proved it in many cases and it was recently completed in residue characteristic not 2 by Waldspurger [**Wa2**].

The conjecture also makes sense in the Archimedean case and was proved by Howe (we take up the Archimedean case in the next section).

Though the conjecture is technically speaking false for finite fields (as was observed by Howe), it should essentially be true there too (this statement can be made more precise).

It is a theorem of Kudla [**Ku1**] that $\sigma_0(\pi)$ has finite length and that if π is supercuspidal then $\sigma_o(\pi)$ is irreducible.

The following lemma is very useful in the study of Howe correspondence, i.e. the correspondence given by conjecture 2.

LEMMA 4.4. *For a dual reductive pair (G, G') in $Sp(W)$, the metaplectic covering $\widetilde{Sp}_\psi(W)$ splits on G unless (G, G') is the pair $(Sp(U), O(V))$ in $Sp(U \otimes V)$, with* $\dim V$ *odd.*

REMARK 4.5. The Lemma 4.4 is not true for the two sheeted cover of the symplectic group in place of $\widetilde{Sp}_\psi(W)$.

4.6 Examples of the Howe Correspondence. We will be looking at certain examples of the Howe correspondence for the dual pair $(Sp(W), O(V)) \subset Sp(W \otimes V)$, mostly for $\dim W = 2$ in which case $Sp(W) \cong SL(2, k)$.

4.6.1. $\dim V = 1, q(x) = ax^2$. Therefore $O(V) = \{\pm 1\}$ and has two representations. Both of these appear in the Weil representation and the corresponding representation of $\widehat{SL}(2, k)$ are on the even and odd functions on k. The representation of $\widehat{SL}(2, k)$ on odd functions is supercuspidal, cf. [**Ge1**], thm.5.19(c).

4.6.2. $\dim V = 2, q(x) = a \cdot$ (norm form of a quadratic field extension K of k). In this case $SO(V) \cong K^1 =$ norm one elements of K, and $O(V)$ is the semi-direct product of K^1 with a group of order 2 acting on K^1 by $x \to \bar{x} = x^{-1}$. Therefore the representations of $O(V)$ are constructed from the characters of K^1 in the following way:

(a) For a character ρ of K^1 such that $\rho^2 \neq 1, \mathrm{Ind}_{K^1}^{O(V)} \rho$ is an irreducible two dimensional representation of $O(V)$. Moreover the representations of $O(V)$ corresponding to characters ρ and ρ^{-1} are the same.

(b) A character ρ of K^1 with $\rho^2 = 1$ extends to a character of $O(V)$ in exactly two ways.

The Weil representation of $Sp(W \otimes V)$ can be realised on $\mathcal{S}(V)$, and by Remark 2.9, the restriction of the Weil representation of $Sp(W \otimes V)$ to $O(V)$ is the natural action of $O(V)$ on $\mathcal{S}(V)$. From this it is easy to see that $R_\psi(O(V))$ consists of all the representations of $O(V)$ except the non-trivial representation of $O(V)$ which is trivial on $SO(V)$. If the norm form of K is $x^2 + dy^2$, then the Weil representation of $Sp(W \otimes V)$ restricted to $Sp(W) = SL_2(k)$ is, by Remark 2.8, the tensor product of the Weil representations of $\widetilde{SL}_2(k)$ associated to the characters ψ_a and ψ_{ad}. The Howe correspondence is therefore a way of parametrising the irreducible constituents of this tensor product in terms of characters of K^1. This example is originally due to Shalika and Tannaka.

4.6.3. $\dim V = 3, q(x) = a \cdot$ (norm form on trace zero elements of a quaternion algebra D). In this case $O(V) \cong SO(V) \times \{\pm 1\}$, and $SO(V) \cong D^*/k^*$, and the Howe correspondence is between representations of $\widehat{SL}(2, k)$ and $D^*/k^* \times \{\pm 1\}$ which is the local analogue of the Shimura correspondence. In this case we have

the following results.

(a) In the case when D a quaternion division algebra, $R_\psi(D^*/k^* \times \{\pm 1\})$ is a set of representations of $D^*/k^* \times \{\pm 1\}$ which goes bijectively to the set of all irreducible representations of D^*/k^* under the restriction map. The set $R_\psi(\widehat{SL}(2,k))$ consists of all genuine representations of $\widehat{SL}(2,k)$ (i.e. not factoring through $SL(2,k)$) not accounted for by the Weil representation on odd functions in 4.6.1 and which are square integrable but do not have a ψ-Whittaker model.

(b) $D = M(2,k)$. In this case $R_\psi(PGL(2,k) \times \{\pm 1\})$ is a set of representations of $PGL(2,k) \times \{\pm 1\}$ which goes bijectively to the set of all irreducible infinite dimensional representations of $PGL(2,k)$ under the restriction map, and $R_\psi(\widehat{SL}(2,k))$ is the set of all genuine representations of $\widehat{SL}(2,k)$ which have a ψ-Whittaker model.

PROOF. Here we prove the statement about $R_\psi(D^*/k^* \times \{\pm 1\})$ only. Analogous proof can also be given in the case of $D = M(2,k)$ but in this case there are some additional problems due to non-compactness, see [**R-S**].

The 3-dimensional representation of $O(V) \cong D_k^*/k^* \times \{\pm 1\}$ can be identified to the representation of $D_k^*/k^* \times \{\pm 1\}$ on the Schwartz space of functions on the vector space of trace 0 elements of D_k on which D_k^* acts by conjugation, and $\{\pm 1\}$ by multiplication.

For a quadratic field K/k, let NK^* be the normaliser of K^* in D_k^*, and let ϕ_K be the character of order 2 of NK^* which is trivial on K^*. Let H_K be the index-2 subgroup of $NK^* \times \{\pm 1\}$ given by the kernel of the homomorphism $\phi_K \times \{\pm 1\}$ to $\mathbb{C}^*$.

The orbits for the action of $D_k^*/k^* \times \{\pm 1\}$ on the trace zero elements of D_k are, besides the origin, of the form $[D_k^*/k^* \times \{\pm 1\}]/H_K$ for a quadratic field extension K/k. It follows from the Mackey theory that the only representations of $O(V)$ appearing in $\mathcal{S}(V)$ are those representations of $D_k^*/k^* \times \{\pm 1\}$ which are trivial on H_K for some quadratic extension K/k. Therefore to prove that the map taking an irreducible representation of $D_k^*/k^* \times \{\pm 1\}$ appearing in $\mathcal{S}(V)$ to its restriction to D_k^*/k^* is an injection, it suffices to prove that for any two quadratic extensions K and L (not necessarily different) of k

$$\mathrm{Hom}_{D_k^*}[\mathrm{Ind}_{NK^*}^{D_k^*} \mathbf{1}, \mathrm{Ind}_{NL^*}^{D_k^*} \phi_L] = 0.$$

We therefore have to prove that there are no non-zero functions f on D_k^* such that

$$f(agb) = \phi_L(b) f(g) \qquad \forall a \in NK^*, g \in D_k^*, b \in NL^*.$$

To prove this, let j_K (resp. j_L) be an element of NK^* (resp. NL^*) which is not in K (resp. L), and observe that for any $g \in D_k^*$, $j_K \cdot K \cap g(j_L \cdot L)g^{-1} \neq 0$. This is because $j_K \cdot K$ and $g(j_L \cdot L)g^{-1}$ are 2-dimensional k-vector spaces, contained in the 3-dimensional space of trace zero elements of D_k. Therefore for all $g \in D_k^*$, there exists $k' \in j_K \cdot K, l' \in j_L \cdot L$ such that $gl'g^{-1} = k'$, or $g = k'gl'^{-1}$. Therefore

$$f(g) = f(k'gl'^{-1}) = \phi_L(l'^{-1}) f(g) = -f(g) = 0.$$

Similarly to prove that the map taking an irreducible representation of $D_k^*/k^* \times \{\pm 1\}$ appearing in $\mathcal{S}(V)$ to its restriction to D_k^*/k^* is a surjection, it suffices to prove that given any irreducible representation of D_k^*, there exists a quadratic field

extension K/k such that the representation has a K^*-invariant vector. This is a well-known fact about representations of quaternion division algebras.

4.6.4. $\dim V = 4$. Suppose that V does not represent any zeros. In that case

$$SO(V) = [D^* \times D^*]^1 = \{(d_1, d_2) \in D^* \times D^* | Nd_1 = Nd_2\},$$

where D is the unique quaternion division algebra over k, which can be identified to V as a quadratic space over k (with the quadratic form on V getting identified to the norm form on D), and the action of $(d_1, d_2) \in [D^* \times D^*]^1$ is by $x \to d_1 x d_2^{-1}$. Let $GSO(V) = D^* \times D^*$, with the natural action on D. Clearly there are exactly 2 orbits for the action of $GSO(V) = D^* \times D^*$ on D, one consisting of the origin only, and the other all of D^*. Therefore from the Mackey theory, the only representations of $D^* \times D^*$ appearing in its natural action on $\mathcal{S}(D)$ are of the type $W \otimes W^*$ for W an irreducible representation of D^*. The representation $W \otimes W^*$ of $GSO(V)$ extends in two ways to a representation of $GO(V)$, the group of orthogonal similitudes, and exactly one of these appears in $\mathcal{S}(D)$. The representations of $O(V)$ appearing in $\mathcal{S}(D)$ are the restrictions of these representations. To conclude, for every irreducible representation of D^*, one can associate a sum of representations of $O(V)$ each of which occurs in the Howe correspondence with SL_2, and conversely every representation of $O(V)$ which occurs in the Howe correspondence with SL_2 occurs in this way.

4.6.5. The Howe duality conjecture is specially simple to state (though not to prove!) for a dual reductive pair of type 2. Take for example the dual pair $(GL(n,k), GL(m,k))$. In this case one is looking at the representation of $GL(n,k) \times GL(m,k)$ on $\mathcal{S}(M(n,m;k))$, where $M(n,m;k)$ is the space of $n \times m$ matrices over k with the action of $(A,B) \in GL(n,k) \times GL(m,k)$ on $X \in M(n,m;k)$ given by AX^tB (this is the restriction of the metaplectic representation of $Sp(nm,k)$ to $GL(n,k) \times GL(m,k)$ up to twisting by a character). The Howe duality conjecture in this case states that given an irreducible representation π of $GL(n,k)$, there exists a unique irreducible representation π' of $GL(m,k)$ such that $\pi \otimes \pi'$ is a quotient of the $GL(n,k) \times GL(m,k)$ module $\mathcal{S}(M(n,m;k))$. In this case a very explicit form of the Howe duality conjecture is expected, which as far as this author could find out, is not known. If $n \leq m$, and if σ_π and $\sigma_{\pi'}$ are the representations of the Weil-Deligne group associated by the Langlands correspondence to π and π' respectively, then one expects that $\sigma_{\pi'} = \nu^{(m-n)/2}\sigma_\pi^* \oplus \nu^{-n/2}$, where ν is the character $|x|$ on k^*. In particular, if $m = n$, one expects that the Howe correspondence is simply taking the dual representation.

REMARK 4.7. Howe correspondence for the case when $\dim V = 3$ produces a correspondence between representations of D^*/k^*, for D a quaternion division algebra, and representations of $\widehat{SL}(2,k)$, and also between square integrable representations of $PGL(2,k)$ and representations of $\widehat{SL}(2,k)$. On the other hand one has the Jacquet-Langlands correspondence between representations of D^*/k^* and $PGL(2,k)$ (which does not depend on the additive character ψ). Therefore given a finite dimensional representation of D^*/k^* one can construct representations of $\widehat{SL}(2,k)$ in two different ways: either directly using Howe correspondence or first using Jacquet-Langlands correspondence and then Howe. It is a basic observation of Waldspurger that these two representations of $\widehat{SL}(2,k)$ are different (as the first one does not have a ψ-Whittaker model whereas the second one has).

REMARK 4.8. We saw in Example 4.6.2 above that for $\dim V = 2, R_\psi(O(V))$ consists of all the representations of $O(V)$ except the non-trivial representation of $O(V)$ which is trivial on $SO(V)$. The Howe lift of this representation to $Sp(W)$ for $\dim W = 4$ is a non-zero super-cuspidal representation, and is related to Srinivasan's representation θ_{10} of $Sp_4(\mathbb{F}_q)$. This representation occurs as the local component of a cuspidal automorphic representation on Sp_4 which contradicts Ramanujan conjecture on Sp_4.

REMARK 4.9. As in the previous remark, one can more generally ask how does the Howe lift vary as we fix the orthogonal group $O(V)$ but change the symplectic group $Sp(W)$. It is a theorem of Kudla [**Ku1**] that if we start with a supercuspidal representation of $O(V)$ then there is a symplectic space W_0 such that the Howe lift to $Sp(W_0)$ is non-zero supercuspidal and the Howe lift to all symplectic spaces of smaller dimensions are zero. Moreover, for a symplectic space W with $\dim W > \dim W_0$, the Howe lift to $Sp(W)$ is non-zero and non-supercuspidal and can be described explicitly in terms of parabolic induction from the Howe lift to $Sp(W_0)$. Kudla has a similar theorem when one fixes W but instead changes V in the same Witt class. Both results are local analogues of Rallis' global theory of towers of theta series liftings [**Ra1**].

5. Howe conjecture in the Archimedean case

The Howe conjecture makes sense in the Archimedean case too. In this case one works with the Harish-Chandra module of the unitary metaplectic representation defined on square integrable functions on a maximal totally isotropic subspace by formulae exactly similar to the one in the non-Archimedean case. We give below the Harish-Chandra module in the Fock model which is more convenient to work with. We will work with the two sheeted covering $\widehat{Sp}(n,\mathbb{R})$ of $Sp(n,\mathbb{R})$ and the inverse image $\widehat{U}(n)$ of $U(n) \subset Sp(n,\mathbb{R})$ as the maximal compact subgroup of $\widehat{Sp}(n,\mathbb{R})$. Let $\mathfrak{sp}$ be the Lie algebra of $\widehat{Sp}(n,\mathbb{R})$ and $\mathfrak{u}$, the Lie algebra of $U(n)$. One has the Cartan decomposition

$$\mathfrak{sp} = \mathfrak{u} \oplus \mathfrak{p} = \mathfrak{sp}^{1,1} \oplus \mathfrak{sp}^{2,0} \oplus \mathfrak{sp}^{0,2},$$

with $\mathfrak{u} = \mathfrak{sp}^{1,1}$. The Fock model of the Weil representation of $(\mathfrak{sp}, \widehat{U}(n))$ is realised on the space $\mathcal{S}$ of polynomial functions on $\mathbb{C}^n$, with representations of $\mathfrak{u}, \mathfrak{p}^{2,0}, \mathfrak{p}^{0,2}$ given as follows.

(1) $\mathfrak{u}$ operates via the differential operators

$$z_i \frac{\partial}{\partial z_j} + \frac{1}{2}\delta_{ij},\ 1 \le i,j \le n,$$

(2) $\mathfrak{p}^{2,0}$ operates by multiplication by $z_i z_j, 1 \le i,j \le n$, and
(3) $\mathfrak{p}^{0,2}$ operates via the differential operators

$$\frac{\partial^2}{\partial z_i \partial z_j},\ 1 \le i,j \le n.$$

The action of $\widehat{U}(n)$ is the standard representation of $U(n)$ on polynomials on $\mathbb{C}^n$ twisted by a character of $\widehat{U}(n)$ which does not factor through $U(n)$.

Let now (G,G') be a dual reductive pair in $Sp(n)$ with maximal compact subgroup K in $\widehat{G}$ and K' in $\widehat{G}'$ such that $K \cdot K'$ is contained in $\widehat{U}(n)$, and let $\mathfrak{g}$ be

the Lie algebra of G, and $\mathfrak{g}'$ of G'. For any irreducible, admissible $(\mathfrak{g}, K)$-module (π, V_π), let $\mathcal{S}(\pi)$ and $\mathcal{S}[\pi]$ be defined as in the non-Archimedean case but now taking homomorphisms in the sense of $(\mathfrak{g}, K)$-modules. Then Howe proves in [**H2**] that just as in Lemma 4.2,

$$\mathcal{S}[\pi] \cong \pi \otimes \sigma_0(\pi)$$

as $(\mathfrak{g}, K) \times (\mathfrak{g}', K')$-modules for a certain finitely generated representation $\sigma_0(\pi)$ of $(\mathfrak{g}', K')$ on which the centre of the universal enveloping algebra of $\mathfrak{g}'$ acts by a character. Furthermore, the representation $\sigma_0(\pi)$ has a unique irreducible quotient $\sigma(\pi)$, and the mapping $\pi \mapsto \sigma(\pi)$ is a bijective correspondence between the irreducible admissible representation of $(\mathfrak{g}, K)$ occuring in the metaplectic representation and the irreducible admissible representation of $(\mathfrak{g}', K')$ occuring in the metaplectic representation. In fact Howe proves a much more precise statement which we take up now; we will follow his fundamental paper on the subject [**Ho2**] very closely to which the reader is referred to for all the proofs.

REMARK 5.1. The simplest example of Howe duality in the real case occurs for the compact pair $(U(n), U(m))$. In this case the metaplectic representation of $Sp(nm)$ restricted to $U(n) \times U(m)$ is essentially the representation of $U(n) \times U(m)$ on polynomial functions on $M(n, m; \mathbb{C})$ with the obvious action. If $n \leq m$, for any irreducible representation σ of $U(n)$, there is a unique representation τ of $U(m)$ such that $\sigma \otimes \tau$ appears in the space of polynomial functions on $M(n, m; \mathbb{C})$.

For any irreducible representation μ of a compact subgroup L of $\widehat{U}(n)$, let $deg(\mu)$ denote the smallest integer d such that μ appears in the space $\mathcal{S}^d$ of polynomials of degree d under the metaplectic representation defined above.

We now discuss Howe duality in the case when one of the groups, say G, in the dual reductive pair (G, G') is compact. In this case the symmetric space associated to G' is hermitian symmetric, and the Cartan decomposition

$$\mathfrak{sp} = \mathfrak{u} \oplus \mathfrak{p} = \mathfrak{sp}^{1,1} \oplus \mathfrak{sp}^{2,0} \oplus \mathfrak{p}^{0,2}$$

restricts to give the Cartan decomposition

$$\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}' = \mathfrak{k}' \oplus \mathfrak{g}'^{2,0} \oplus \mathfrak{g}'^{0,2}.$$

Let

$$\mathcal{H}(G) = \{P \in \mathcal{S} | X \cdot P = 0 \quad \text{for all} \quad X \in \mathfrak{g}'^{0,2}\}.$$

The vector space $\mathcal{H}(G)$ is called the space of harmonic polynomials for the compact group G. Clearly, $\mathcal{H}(G)$ is invariant under $\widehat{G}$ and K'. With this notation, we have the following theorem due to Howe, cf. [**H2**] and [**H3**].

THEOREM 5.2. *Let (G, G') be a dual reductive pair with G compact, and let σ be a finite dimensional irreducible representation of $\widehat{G}$. Then we have the following.*

(a) *The σ-isotypic part $\mathcal{H}(G)_\sigma$ of $\mathcal{H}(G)$ is precisely the σ-isotypic part in $\mathcal{S}^{\deg(\sigma)}$.*
(b) *The representation $\mathcal{H}(G)_\sigma$ of $\widehat{G} \times K'$ is irreducible.*
(c) *If we write $\mathcal{H}(G)_\sigma = \sigma \otimes \tau'$, for an irreducible representation τ' of K', then $\sigma \mapsto \tau'$ is an injective map from the set of irreducible representations of $\widehat{G}$ appearing in the metaplectic representation to the set of irreducible representations of K'.*
(d) *The σ-isotypic part $\mathcal{S}_\sigma$ of $\mathcal{S}$ is an irreducible representation of $\widehat{G} \times \widehat{G}'$, and is therefore of the form $\sigma \otimes \tau$ for an irreducible representation τ of $\widehat{G}'$, and*

one has

$$\mathcal{S}_\sigma = \mathcal{U}(\mathfrak{g}') \cdot \mathcal{H}(G)_\sigma,$$

where $\mathcal{U}(\mathfrak{g}')$ is the universal enveloping algebra of $\mathfrak{g}'$.

(e) *The representation τ of $\widehat{G}'$ contains the representation τ' of K' on which $\mathfrak{g}'^{0,2}$ acts trivially. Therefore τ is the unique highest weight module of $\mathfrak{g}'$ containing τ' as the highest K'-type.*

This theorem has been used by Howe to prove the general duality for real groups, to which we come later, using the following structural theorem about dual reductive pairs.

THEOREM 5.3. *Let (G, G') be a dual reductive pair in $Sp(n)$ with maximal compact subgroups K and K' respectively, which are assumed to be in $U(n)$. For any subgroup H of $Sp(n)$, let $Z(H)$ denote the centraliser of H in $Sp(n)$, and let $H_u = H \cap U(n)$, which will be assumed to be a maximal compact subgroup of H. Then we have the following.*

(a) *The pair $(K, Z(K))$ is a dual reductive pair, and similarly $(K', Z(K'))$ is a dual reductive pair.*

(b) *The pair $(Z(K)_u, Z(K')_u)$ is a dual reductive pair in which both the groups $Z(K)_u$, and $Z(K')_u$ are products of compact unitary groups.*

EXAMPLE 5.4. For the dual reductive pair $(O(p,q), Sp(n))$ with $pq \neq 0$, the maximal compact subgroup of $O(p,q)$ is $O(p) \times O(q)$ whose centraliser in $Sp(n(p+q))$ is $Sp(n) \times Sp(n)$. The maximal compact subgroup of $Sp(n)$ is $U(n)$ whose centraliser in $Sp(n(p+q))$ is $U(p,q)$.

The Howe duality for real groups has subordinate to it a duality correspondence between the representations of their maximal compact subgroups which appear in the metaplectic representation. Here is the theorem on the correspondence between the irreducible representations of maximal compact subgroups.

THEOREM 5.5. *Let (G, G') be a dual reductive pair with K as a maximal compact subgroup of $\widehat{G}$ and K' as a maximal compact subgroup of $\widehat{G}'$. Then for an irreducible representation σ of K, there exists a unique irreducible representation σ' of K' such that $\mathcal{H}(K)_\sigma \cap \mathcal{H}(K')_{\sigma'} \neq 0$. Moreover, if $\mathcal{H}(K)_\sigma \cap \mathcal{H}(K')_{\sigma'} \neq 0$, then $\mathcal{H}(K)_\sigma \cap \mathcal{H}(K')_{\sigma'} \cong \sigma \otimes \sigma'$ as a $K \times K'$-module.*

REMARK 5.6. In the cases such as $(O(p,q), Sp(n))$ or $(U(p,q), U(r,s))$, where K is isomorphic to $K_1 \times K_2$, and the centraliser of K is isomorphic to $G' \times G'$, with (K_1, G') and (K_2, G') themselves dual reductive pairs, the correspondence of Theorem 5.5 between representations of K and K' can be described in terms of the correspondence of Theorem 5.2(c) as follows. Let $\sigma_1 \otimes \sigma_2$ be a representation of $K = K_1 \times K_2$. If the representations of K' associated by Theorem 5.2(c) to representations σ_1 and σ_2 of K_1 and K_2 are τ_1' and τ_2' respectively, then the representation of K' associated to the representation $\sigma_1 \otimes \sigma_2$ of K is the irreducible representation in the tensor product $\tau_1' \otimes \tau_2'$ whose highest weight is the sum of highest weights of τ_1' and τ_2'.

For an irreducible admissible $(\mathfrak{g}, K)$-module π, let $deg(\pi)$ denote the smallest degree of any K-type whose image is non-zero under the surjection from $\mathcal{S}$ to $\mathcal{S}[\pi]$. Then the following is the more precise version of the Howe correspondence for real groups.

THEOREM 5.7. *Let π be an irreducible admissible representation of $(\mathfrak{g}, K)$, and let μ be a representation of K appearing in π and having degree $deg(\mu) = deg(\pi)$. Then the representation $\sigma(\pi)$ of $(\mathfrak{g}', K')$ obtained by the Howe correspondence from π, contains the representation μ' of K' which is such that $\mathcal{H}(K)_\mu \cap \mathcal{H}(K')_{\mu'} \neq 0$. Moreover, the representation $\sigma(\pi)$ does not contain any representation of K' other than μ' which appears in $\mathcal{H}(K)_\mu$.*

REMARK 5.8. For a dual reductive pair (G, G') in $Sp(n, \mathbb{R})$ with one of G or G' compact, the Howe correspondence has been explicitly worked out by Kashiwara and Vergne in [**K-V**]. In this case, as already noted in theorem 5.2, all the representations are highest weight modules.

6. The spherical case

By an unramified classical group over a non-Archimedean local field k, we will mean one of the following groups.

(a) The group $GL(n, \ell)$ where ℓ is an unramified extension of k.
(b) The unitary group of an ϵ-hermitian space W over an unramified extension ℓ of k, with a lattice $\mathcal{L}$ in W with $\mathcal{L}^\perp = \mathcal{L}$ (for a character ψ of ℓ such that $\psi(\mathcal{O}_\ell) = 1$ but $\psi(\pi^{-1}\mathcal{O}_\ell) \not\equiv 1$).

The subgroup $GL(n, \mathcal{O}_\ell)$ of $GL(n, \ell)$ in case (a), and the stabiliser of the lattice $\mathcal{L}$ in case (b) will be called standard maximal compact subgroups.

For $\mathcal{L} = \sum_{i=1}^n \mathcal{O}_k e_i \oplus \sum_{i=1}^n \mathcal{O}_k f_i$ where $\{e_i, f_i\}$ is a symplectic basis of W, i.e. $\langle e_i, e_j \rangle = 0, \langle f_i, f_j \rangle = 0, \langle e_i, f_j \rangle = \delta_{ij}$, it is clear that $\mathcal{L}^\perp = \mathcal{L}$. Therefore $Sp(W)$ is an unramified classical group, and we let K be the stabiliser of $\mathcal{L}$.

Given a dual reductive pair (G, G') in $Sp(W)$ consisting of unramified pairs, we can assume that G has a standard maximal compact subgroup $\mathcal{K}$, and G' has a standard maximal compact subgroup $\mathcal{K}'$ such that $\mathcal{K} \cdot \mathcal{K}'$ is contained in K.

From section 2.6, we know that the metaplectic covering of $Sp(W)$ splits on K. Therefore $\mathcal{K}$ can be thought of as a subgroup of $\widetilde{G}$, and $\mathcal{K}'$ a subgroup of $\widetilde{G}'$. Let $\mathcal{H}_{\psi^{-1}}(\widetilde{G}//\mathcal{K})$ denote the space of $\mathcal{K}$-bi-invariant functions f on $\widetilde{G}$ which are compactly supported modulo $\mathbb{C}^*$, such that $f(zg) = \psi^{-1}(z)f(g)$ for all $z \in \mathbb{C}^*$. $\mathcal{H}_{\psi^{-1}}(\widetilde{G}//\mathcal{K})$ is an algebra under convolution (the integral in the convolution is over $\widetilde{G}/\mathbb{C}^*$), and operates in a natural way on representations of $\widetilde{G}$ with central character ψ.

THEOREM 6.1(HOWE). *Given a representation π of $\widetilde{G}$ with a $\mathcal{K}$ fixed vector, either there is no representation π' of $\widetilde{G}'$ such that $\pi \otimes \pi'$ is a quotient of the Weil representation $(\omega_\psi, \mathcal{S})$ of $\widetilde{Sp}(W)$, or there is exactly one representation π' of $\widetilde{G}'$ such that $\pi \otimes \pi'$ is a quotient of the Weil representation $(\omega_\psi, \mathcal{S})$ of $\widetilde{Sp}(W)$, and this π' has a $\mathcal{K}'$ fixed vector. Moreover, the algebras $\mathcal{H}_{\psi^{-1}}(\widetilde{G}//\mathcal{K})$, and $\mathcal{H}_{\psi^{-1}}(\widetilde{G}'//\mathcal{K}')$ operate with the same ring of endomorphisms on the space of $\mathcal{K} \cdot \mathcal{K}'$ fixed vectors of $\mathcal{S}$.*

7. Seesaw pairs

We introduce in this section the important concept of seesaw pairs due to Kudla.

DEFINITION 7.1. A pair (G, G') and (H, H') of dual reductive pairs in a symplectic group $Sp(W)$ is called a seesaw pair if $H \subset G$ and $G' \subset H'$.

It is generally pictorially depicted by the diagram

$$\begin{array}{ccc} G & & H' \\ & \bowtie & \\ H & & G' \end{array}$$

where vertical arrows are inclusions and slanted arrows connect members of a dual reductive pair.

EXAMPLES 7.2.

7.2.1.
$$\begin{array}{ccc} Sp(W_1 \oplus W_2) & & O(V) \times O(V) \\ & \bowtie & \\ Sp(W_1) \times Sp(W_2) & & O(V) \end{array}$$

7.2.2.
$$\begin{array}{ccc} O(V_1 \oplus V_2) & & Sp(W) \times Sp(W) \\ & \bowtie & \\ O(V_1) \times O(V_2) & & Sp(W) \end{array}$$

7.2.3.
$$\begin{array}{ccc} U(V_1 \oplus V_2) & & U(W) \times U(W) \\ & \bowtie & \\ U(V_1) \times U(V_2) & & U(W) \end{array}$$

7.2.4.
$$\begin{array}{ccc} Sp(W_1 \oplus W_1^*) & & GL(V) \\ & \bowtie & \\ GL(W_1) & & O(V). \end{array}$$

Let (G, G') and (H, H') be a pair of dual reductive pairs in $Sp(W)$ forming a seesaw pair:

$$\begin{array}{ccc} G & & H' \\ & \bowtie & \\ H & & G' \end{array}.$$

Let π_H be in $R_\psi(H)$ and $\pi_{G'}$ in $R_\psi(G')$, and recall the notation $\sigma_0(\pi_H)$ and $\sigma_0(\pi_{G'})$ introduced before Conjecture 2.

LEMMA 7.3. *With notation as above,*

$$\mathrm{Hom}_H[\sigma_0(\pi_{G'}), \pi_H] = \mathrm{Hom}_{G'}[\sigma_0(\pi_H), \pi_{G'}].$$

PROOF. Let the Weil Representation of $Sp(W)$ be realised on $\mathcal{S}$. Clearly,

$$\begin{aligned} \mathrm{Hom}_{H \times G'}[\mathcal{S}, \pi_H \otimes \pi_{G'}] &= \mathrm{Hom}_{H \times G'}[\sigma_0(\pi'_G) \otimes \pi_{G'}, \pi_H \otimes \pi_{G'}] \\ &= \mathrm{Hom}_H[\sigma_0(\pi_{G'}), \pi_H]. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{Hom}_{H \times G'}[\mathcal{S}, \pi_H \otimes \pi_{G'}] &= \mathrm{Hom}_{H \times G'}[\pi_H \otimes \sigma_0(\pi_H), \pi_H \otimes \pi_{G'}] \\ &= \mathrm{Hom}_{G'}[\sigma_0(\pi_H), \pi_{G'}]. \end{aligned}$$

This proves the lemma.

REMARK 7.4. To see how seesaw pairs are used in practice, let us look at the following seesaw pair in the real case

$$\begin{array}{ccc} O(p,q) & & Sp(V)\times Sp(V) \\ & \bowtie & \\ O(p)\times O(q) & & Sp(V). \end{array}$$

Let σ be a representation of $Sp(V)$, and $\tau_1 \otimes \tau_2$ of $O(p) \times O(q)$. As remarked earlier, since $O(p)$ and $O(q)$ are compact groups, the Howe lifts of τ_1 and τ_2 are known by [**K-V**]. By Lemma 7.3, $\sigma_0(\sigma)$ contains $\tau_1 \otimes \tau_2$ if and only if σ is a quotient of $\sigma_0(\tau_1) \otimes \sigma_0(\tau_2)$. This relates the Howe lifting problem to a problem about tensor product of representations, and information about either one therefore gives information about the other one.

Here is an example from [**HK**] who use the following seesaw pair to give the decomposition of a discrete series representation of $Sp(2)$ restricted to $Sp(1)\times Sp(1)$.

$$\begin{array}{ccc} Sp(2) & & O(2,2)\times O(2,2) \\ & \bowtie & \\ Sp(1)\times Sp(1) & & O(2,2) \end{array}.$$

Towards the analysis by Harris and Kudla on the decomposition of a discrete series representation of $Sp(2)$ restricted to $Sp(1) \times Sp(1)$, we simply note here that the Howe correspondence between an orthogonal group and a symplectic group is known by the work of Adams [**Ad1**], and one can calculate the tensor product of two representations of $O(2,2)$ by the work of Repka [**Re**].

8. The theta correspondence

Let k be a number field, and $\mathbb{A}$ the adele-ring of k. For G an algebraic group over k, let $G(k)$ denote the group of k-rational points of G, and let $G(\mathbb{A})$ denote the adelic points of G. Let W be a symplectic vector space over k, and $H(W)$ the associated Heisenberg group which is an algebraic group over k.

Let $W = W_1 \oplus W_2$ be a complete polarisation of W over k. Let $\mathcal{S}(W_1(\mathbb{A}))$ denote the space of Schwartz-Bruhat functions on $W_1(\mathbb{A})$. Given a character $\psi : \mathbb{A}/k \to \mathbb{C}^*$, one can construct a representation of $H(W)(\mathbb{A})$ on $\mathcal{S}(W_1(\mathbb{A}))$ on which the centre of $H(W)(\mathbb{A})$ (which is $\mathbb{A}$) operates via ψ:

$$\begin{aligned} \rho_\psi(w_1)f(x) &= f(x+w_1) && \text{for all} \quad x, w_1 \in W_1(\mathbb{A}), \\ \rho_\psi(w_2)f(x) &= \psi(\langle x, w_2\rangle)f(x) && \text{for all} \quad x \in W_1(\mathbb{A}), w_2 \in W_2(\mathbb{A}), \\ \rho_\psi(t)f(x) &= \psi(t)f(x) && \text{for all} \quad t \in \mathbb{A}, x \in W_1(\mathbb{A}). \end{aligned}$$

Define a linear functional Θ on $\mathcal{S}(W_1(\mathbb{A}))$ by

$$\Theta(\phi) = \sum_{x \in W_1(k)} \phi(x) \quad \text{for} \quad \phi \in \mathcal{S}(W_1(\mathbb{A})).$$

The following lemma is easy to prove.

LEMMA 8.1. *The series defining $\Theta(\phi)$ converges absolutely, and $\phi \to \Theta(\phi)$ defines an $H(W)(k)$-invariant linear form on $\mathcal{S}(W_1(\mathbb{A}))$.*

The following basic theorem is due to Weil [**We**].

THEOREM 8.2. *The space of $H(W)(k)$-invariant forms on $\mathcal{S}(W_1(\mathbb{A}))$ is generated by Θ.*

As in the local case, $Sp(W)(\mathbb{A})$ operates on $H(W)(\mathbb{A})$, and by the uniqueness of the representation of $H(W)(\mathbb{A})$ with central character ψ, we obtain a projective representation of $Sp(W)(\mathbb{A})$ via intertwining operators. This gives a $\mathbb{C}^*$-covering of $Sp(W)(\mathbb{A})$, denoted by $\widetilde{Sp}(W)(\mathbb{A})$, such that for each place v of k one has the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \widetilde{Sp}(W)(k_v) & \longrightarrow & Sp(W)(k_v) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \widetilde{Sp}(W)(\mathbb{A}) & \longrightarrow & Sp(W)(\mathbb{A}) & \longrightarrow & 0. \end{array}$$

We note that $\widetilde{Sp}(W)(\mathbb{A})$ is not the adelic points of an algebraic group over k, so the notation is not very appropriate but is customary. The global metaplectic group comes equipped with a representation on $\mathcal{S}(W_1(\mathbb{A}))$, called the global metaplectic or Weil representation.

For any k-algebraic subgroup H of $Sp(W)$, let $\widetilde{H}(\mathbb{A})$ denote the inverse image of $H(\mathbb{A})$ in $\widetilde{Sp}(W)(\mathbb{A})$, and let $\widetilde{H}(k)$ denote the inverse image of $H(k)$ in $\widetilde{Sp}(W)(\mathbb{A})$.

Since there is an action of the semi-direct product $H(W)(k) \rtimes \widetilde{Sp}(W)(k)$ on $\mathcal{S}(W_1(\mathbb{A}))$, it is clear that for all $g \in \widetilde{Sp}(W)(k)$, $\phi \mapsto \Theta(g\phi)$ is also an $H(W)(k)$-invariant form on $\mathcal{S}(W_1(\mathbb{A}))$. Therefore by the uniqueness Theorem 8.2 above, $\Theta(g\phi) = \lambda_g \Theta(\phi)$ for some $\lambda_g \in \mathbb{C}^*$. Clearly $g \to \lambda_g$ is a character on $\widetilde{Sp}(W)(k)$, which gives the unique splitting of the exact sequence

$$0 \to \mathbb{C}^* \to \widetilde{Sp}(W)(k) \to Sp(W)(k) \to 0.$$

Because of this splitting, we can think of $Sp(W)(k)$ as a subgroup of $\widetilde{Sp}(W)(\mathbb{A})$, and for any k-algebraic subgroup H of $Sp(W)$, $H(k)$ can also be thought of as a subgroup of $\widetilde{H}(\mathbb{A})$. The reader must be cautioned here that if the global metaplectic covering of $Sp(W)(\mathbb{A})$ splits on $H(\mathbb{A})$, it is not a priori clear that one can choose a splitting such that the image of $H(\mathbb{A})$ contains $H(k)$. The problem arises of course because there may not be a unique splitting of $H(\mathbb{A})$; see [**Ge2**] for H a unitary group.

Given any function $\phi \in \mathcal{S}(W_1(\mathbb{A}))$, the following formula defines a function θ_ϕ on $Sp(W)(k) \setminus \widetilde{Sp}(W)(\mathbb{A})$

$$\theta_\phi(g) = \Theta(g \cdot \phi).$$

The function θ_ϕ is known to be an automorphic function on $Sp(W(k)) \setminus \widetilde{Sp}(W)(\mathbb{A})$ (i.e., it is a slowly increasing, K-finite, Z-finite function where Z is the centre of the universal enveloping algebra of $Sp(W)(\mathbb{R})$). It has the property that $\theta_\phi(hg) = \theta_{h\phi}(g)$.

We are now ready to define the so-called theta correspondence which defines a correspondence between automorphic forms on one member of a dual reductive pair to those of the other member of the dual reductive pair.

Let (G, G') be a pair of algebraic groups over k which form a dual reductive pair in $Sp(W)$. Restricting the function θ_ϕ on $\widetilde{Sp}(W)(\mathbb{A})$ to $\widetilde{G}(\mathbb{A}) \times \widetilde{G'}(\mathbb{A})$, we get a function $\theta_\phi(g, g')$ on $\widetilde{G}(\mathbb{A}) \times \widetilde{G'}(\mathbb{A})$, called the theta-kernel. Now let $\mathcal{A}$ be

an irreducible space of cusp forms on $G(k) \setminus \widetilde{G}(\mathbb{A})$. For $f \in \mathcal{A}$, define a function $\theta_\phi(f)(g')$ on $G'(k) \setminus \widetilde{G}'(\mathbb{A})$ by

$$\theta_\phi(f)(g') = \int_{G(k)\setminus\widetilde{G}(\mathbb{A})} \theta_\phi(g, g') f(g) dg.$$

The span of $\theta_\phi(f)$ where ϕ runs over $\mathcal{S}(W_1(\mathbb{A}))$ is a $\widetilde{G}'(\mathbb{A})$-invariant space of functions. From the property $\theta_\phi(hg) = \theta_{h\phi}(g)$ noted above, it follows that if f_1 and f_2 belong to the same irreducible representation of $\widetilde{G}(\mathbb{A})$, the span of $\theta_\phi(f_1)$ and $\theta_\phi(f_2)$ are the same. The span of $\theta_\phi(f)$ as f runs over $\mathcal{A}$, and ϕ runs over $\mathcal{S}(W_1(\mathbb{A}))$ is called the θ-lift of the cuspidal representation $\mathcal{A}$, and will be denoted by $\Theta(\mathcal{A}, \psi)$. This space is not necessarily irreducible, and any irreducible sub-quotient of it is called a representation obtained by theta correspondence.

8.3 Examples of the theta Correspondence. We will be looking at certain examples of the theta correspondence for the dual pair $(Sp(W), O(V)) \subset Sp(W \otimes V)$, when $\dim W = 2$ in which case $Sp(W) \cong SL(2, k)$.

8.3.1. $\dim V = 1, q(x) = ax^2$, $k = \mathbb{Q}$. Automorphic forms $f_S = \prod f_v$ on $O(V)$ correspond bijectively to finite subsets S of even cardinality of the set of places of $\mathbb{Q}$ such that $f_v(-1) = -1$ if and only if $v \in S$. Theta lift of f_S is a modular form of weight $\frac{1}{2}$ or $\frac{3}{2}$ depending on whether $\infty \notin S$, or $\infty \in S$. If S is the empty set, then the theta lift of f_S for an appropriate choice of the function ϕ is the usual theta function on the upper half plane: $\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$. If S is not empty, the theta lift of f_S is a cusp form. See [**Ge1**] for details on all this.

8.3.2. $\dim V = 2, q(x) = a \cdot$ (norm form of a quadratic field extension K of k). An automorphic form on $O(V)$ is given by a grossencharacter on the idele class group J_K/K^*, and the theta lift constructs an automorphic form on $GL_2(k)$. This construction in the case of $k = \mathbb{Q}$ goes back to Hecke and Maass.

8.3.3. $\dim V = 3, q(x, y, z) = x^2 - yz, SO(V) \cong PGL_2$. The theta lift in this case from $\widetilde{SL}_2$ to PGL_2 is the Shimura lifting which associates a modular form of weight $n - 1$ to a modular form of half-integral weight $\frac{n}{2}$.

8.3.4. $\dim V = 4$. If V is anisotropic over k, then the theta lifting is related to the Jacquet-Langlands correspondence which constructs automorphic forms on GL_2 from automorphic forms on quaternion division algebras.

If V has Witt index 1, then there is a unique quadratic extension K/k over which V splits. In this case the theta lifting is the base change from GL_2 over k to GL_2 over K. If $k = \mathbb{Q}$, and K is a real quadratic field, it is the Doi-Naganuma lift.

8.4 Global Howe Conjecture. Let (G, G') be a dual reductive pair and $\pi = \otimes\pi_v$, a representation of $\widetilde{G}(\mathbb{A})$. One knows that at almost all the places v of k, the group G is an unramified classical group and the representation π_v has a fixed vector under a standard maximal compact subgroup of $G(k_v)$. Therefore from Theorem 6.1, if the Howe lift $\sigma(\pi_v)$ of π_v is non-zero, it also has a fixed vector under a standard maximal compact subgroup of $G'(k_v)$. We can therefore form the restricted direct product of the representations $\sigma(\pi_v)$, to be denoted by $\sigma(\pi)$. Howe had conjectured that if π is an automorphic representation of $\widetilde{G}$ then $\sigma(\pi)$ is also automorphic. This turned out to be false as was shown by Piatetski-Shapiro in [**Ps**], following the work of Waldspurger. One hopes that this conjecture, though not entirely correct, is basically correct. Possibly, the problems are related to L-indistiguishability and the counter-example of Piatetski-Shapiro suggests that even

though $\sigma(\pi)$ may not be automorphic, there may be a representation $\sigma'(\pi)$ which differs from $\sigma(\pi)$ in only finitely many places and is automorphic.

On the positive side, we have the following (folklore) theorem for which we refer to [**GRS**] for the proof.

THEOREM 8.5. *Let (G, G') be a dual reductive pair and $\pi = \otimes\pi_v$, a cuspidal automorphic representation of $\widetilde{G}$ such that $\Theta(\pi, \psi)$ consists of cusp forms on $\widetilde{G}'(\mathbb{A})$. Then*

(i) *$\Theta(\pi, \psi)$ is an irreducible representation $\pi' = \otimes\pi'_v$ of $\widetilde{G}'(\mathbb{A})$.*
(ii) *All the local representations π'_v are the Howe lifts of π_v (for the character ψ_v); in particular the global Howe conjecture is true in this case.*
(iii) *The theta lift $\Theta(\pi', \psi)$ of π' is non-zero. If it is cuspidal, it is equivalent to π.*

REMARK 8.6. To see a situation where $\Theta(\pi, \psi)$ is known to consist only of cusp forms, we recall a theorem of Rallis [**Ra1**] (see also [**H-Ps**]) that for the dual pair $(O(V), Sp(n))$, π a cuspidal automorphic representation on $O(V)$, $\Theta(\pi, \psi)$ consists only of cusp forms if and only if the theta lift of π to $Sp(n-1)$ is zero for the dual pair $(O(V), Sp(n-1))$.

The following fundamental theorem of Waldspurger [**Wa1**] gives criterion for the non-vanishing of theta lifts in the simplest situation of the dual reductive pair $(O(2,1), \widetilde{SL}_2)$. As we saw in 4.6.3, there is no loss of information in going from $O(2,1)$ to $SO(2,1) = PGL_2$, and this is what Waldspurger considers: the pair $(PGL_2, \widetilde{SL}_2)$.

THEOREM 8.7.

(a) *The theta lift of an automorphic cuspidal representation π of PGL_2 to $\widetilde{SL}_2$ is non-zero if and only if $L(\pi, \frac{1}{2}) \neq 0$.*
(b) *The theta lift of an automorphic cuspidal representation π of PGL_2 to $\widetilde{SL}_2$ is orthogonal to all the automorphic forms on $\widetilde{SL}_2$ arising from the pair $(O(1), \widetilde{SL}_2)$ (and for any choice of the additive character ψ) where $O(1)$ is the orthogonal group in one variable.*
(c) *Let σ be a cuspidal automorphic representation of $\widetilde{SL}_2$ orthogonal to all the automorphic forms on $\widetilde{SL}_2$ arising from the pair $(O(1), \widetilde{SL}_2)$ (and for any choice of the additive character ψ), then the theta lift of σ to PGL_2 is an irreducible cuspidal representation which is non-zero if and only if σ has a Whittaker model for ψ^{-1}. Moreover, the condition that σ has a Whittaker model can be interpreted as a non-vanishing of a certain L-function at $\frac{1}{2}$.*

We now come to the global analogue of Lemma 7.3 about seesaw pairs. Before we state it, let us introduce the notation $\langle f_1, f_2 \rangle_E = \int_{\mathbb{C}^* E(k) \backslash \widetilde{E}(\mathbb{A})} f_1 \bar{f}_2 de$ where E is a k-algebraic subgroup of $Sp(W)$, and f_1, f_2 are functions on $E(k) \backslash \widetilde{E}(\mathbb{A})$ with the same unitary central characters. Let

$$\begin{array}{ccc} G & & H \\ & \bowtie & \\ H' & & G' \end{array}$$

be a seesaw pair. The proof of the following analogue of Lemma 7.3 is clear and will be omitted.

LEMMA 8.8. *For f_1 a cusp form on H' and f_2 a cusp form on G', and an arbitrary $\phi \in \mathcal{S}(W_1(\mathbb{A}))$, we have*

$$\langle \theta_\phi(f_1), f_2 \rangle_{G'} = \langle f_1, \bar{\theta}_\phi(f_2) \rangle_{H'},$$

where $\theta_\phi(f_1)$ which is a function on $\widetilde{H}(\mathbb{A})$ has been restricted to $\widetilde{G}'(\mathbb{A})$, similarly $\theta_\phi(f_2)$ which is a function on $\widetilde{G}(\mathbb{A})$ has been restricted to $\widetilde{H}'(\mathbb{A})$.

This is an identity between inner products of automorphic forms on different groups, and is often useful to relate L-functions of automorphic forms to certain period integrals involving Eisenstein series (which arise as theta lifts of certain automorphic forms by the Siegel-Weil theorem). We do not go into the details on this but refer to [**Ku2**] for several examples.

9. Questions

Here we want to point out some of the main open problems of the theory that we have been discussing.

9.1 Local question. As the Howe conjecture is now basically proved (except in residue characteristic 2 case), the question arises what exactly is the Howe correspondence in terms of familiar parametrisations of representations. For example, one would like to understand the Howe correspondence in terms of Langlands' functoriality. More precisely, let (G, G') be a dual reductive pair and assume for simplicity that the metaplectic covering of the corresponding symplectic group splits over $G \times G'$. Let π be an irreducible representation of G, and $\sigma(\pi)$ the Howe lift of π (which of course depends on the additive character). Then the question is whether there is a mapping $\rho_\pi : {}^L G \to {}^L G'$ such that the representation $\sigma(\pi)$ of G' lies in the L-packet of the lift of the representation π of G under this map. Of course for a general pair (G, G'), there may not be any non-trivial maps of ${}^L G$ to ${}^L G'$, so this naive question is meaningless but one might hope that there are non-trivial maps from endoscopic subgroups ${}^L H$ of ${}^L G$ to ${}^L G'$, and the representation π comes from an endoscopic lift of a representation of H. So we can modify our naive question to ask that if $\sigma(\pi)$ is non-zero, whether there is an endoscopic subgroup ${}^L H_\pi$ of ${}^L G$ (more precisely an endoscopic data) such that the representation π is an endoscopic lift of a representation of H_π with Langlands parameter σ_π with values in ${}^L H_\pi$, and a mapping $\rho_\pi : {}^L H_\pi \to {}^L G'$ such that $\sigma(\pi)$ lies in the L-packet of $\rho_\pi(\sigma_\pi)$. Even this conjecture seems to be false as two representations of G which are in the same L-packet might lift to representations of G' which are not in the same L-packet. But the question is expected to have an affirmative answer if one replaces L-packets by the bigger Arthur packets. A related question is to understand how does the Howe correspondence change when one replaces G or G' by groups which are inner forms of these. See the work of J. Adams in [**Ad2**] for precise conjectures and results in the real case. See also the work of Rallis [**Ra2**] and the paper of Gelbart in this volume [**Ge2**] concerning this question. For the dual pair $(GL(n), GL(m))$, see 4.6.5 for what is expected.

9.2 Global question. Let k be a global field, (G, G') a dual reductive pair over k, and $\pi = \otimes \pi_v$ a cuspidal automorphic representation of $\widetilde{G}(\mathbb{A})$. It is clear that if $\Theta(\pi, \psi) \not\equiv 0$, then for all the local components π_v of π, the Howe lift (with respect to ψ_v) is non-zero. The main question concerning the global theory is to find necessary and sufficient conditions for $\Theta(\pi, \psi)$ to be non-zero. If we assume that

$\pi = \otimes\pi_v$ is such that the Howe lift of π_v is non-zero for all v, then in many cases there is a condition involving global L-function which controls whether or not there is an automorphic representation $\pi' = \otimes\pi'_v$ in the same global L-packet as π (i.e., π_v and π'_v belong to the same L-packet for all v and are equal for almost all v) whose theta lift is non-zero. The work of Waldspurger [**Wa1**] recalled in Theorem 8.7 above, and the subsequent work of Rallis [**Ra3**] are in this direction. However, in many other cases, there is no such condition involving global L-function and there is an automorphic representation π' in the same global L-packet as π whose theta lift is non-zero as soon as all the local Howe lifts are non-zero. The situation is not clear at the moment. See also the work [**GRS**] where rather complete results have been obtained for the theta lifts of global L-packets for the pair $(U(2,1), U(1,1))$.

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